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THE PENETRATION OF DIFFUSE ULTRAVIOLET RADIATION INTO INTERSTELLAR CLOUDS

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ABSTRACT

We show that the solution of the transfer equation appropriate for models of the penetration of diffuse UV radiation into interstellar clouds, subject to attenuation by coherent, nonconservative, anisotropic scattering from grains, can be expressed analytically, with arbitrary accuracy, by means of the spherical harmonics method. Models of plane-parallel and homogeneous spherical clouds are given as functions of three parameters: τ_c the central optical depth, ω the single scattering albedo, and g the parameter in the Henyey-Greenstein phase function. These models qualitatively confirm the results of earlier Monte Carlo simulations of dust scattering, but reveal quantitative discrepancies: the earlier results overestimated the actual mean intensity, often by more than an order of magnitude.

Subject headings: interstellar: matter — radiative transfer — ultraviolet: general

I. INTRODUCTION

UV radiation fundamentally influences the properties of interstellar clouds by controlling the ionization and dissociation balance of important ions and molecules. By shielding the interior of clouds from ambient, diffuse UV radiation, dust grains create an environment in clouds which favors low temperatures and molecule formation. Although the scattering properties of grains have not been determined definitively, some observations suggest that grains can have high albedos and that the redistribution of scattered light can be far from isotropic (Lillie and Witt 1976). Therefore, an investigation of the penetration of diffuse UV radiation into clouds should consider the effects of coherent, nonconservative, anisotropic scattering by grains.

In detailed calculations of the transfer equation for this problem, Sandell and Mattila (1975) assumed the Henyey-Greenstein phase function and used the technique of Monte Carlo simulation to demonstrate that such scattering properties greatly reduce the shielding effects of grains from the attenuation that would result if interstellar reddening were a purely absorptive process (Stief *et al.* 1972). Using a rather different phase function, Whitworth (1975) also found the shielding to be greatly reduced. In this paper we show that the transfer equation for the penetration of diffuse UV radiation into clouds can be solved with arbitrary accuracy in analytic form by means of the spherical harmonics method (see, e.g., Davison and Sykes 1957).

In § II we solve the radiative-transfer problem by representing the specific intensity as an expansion in Legendre polynomials. The expansion coefficients are functions of optical depth, and are exponential functions in plane-parallel geometry and spherical Bessel functions in homogeneous spherical geometry. These functions exhibit decay rates that are found as roots of a certain characteristic equation. The smallest root determines the asymptotic solution for the mean intensity in clouds of sufficient optical depth. In § III we present a series of models in which the grains scatter according to a Henyey-Greenstein phase function. The Henyey-Greenstein function has become conventional as a simple model to interpret observations of grains and is the form used by Sandell and Mattila in their Monte Carlo models. In § IV we also treat as a special case the simple phase function used by Whitworth (1975). In § V we discuss the accuracy and efficiency of the method.

II. APPLICATION OF THE SPHERICAL HARMONICS METHOD

The spherical harmonics method allows us to generate arbitrarily accurate solutions for the radiative-transfer equation appropriate for coherent, nonconservative, anisotropic scattering of ambient, external UV photons into an interstellar cloud subject to attenuation by embedded dust grains. With slight modification the techniques apply to both plane-parallel and homogeneous spherical clouds. In overview, we represent the specific intensity as an expansion in Legendre polynomials, assuming azimuthal symmetry for both the specific intensity and the phase function for scattered light. The coefficients of the Legendre polynomials are either exponential functions or modified spherical Bessel functions of the first kind, which express the depth dependence in the case of plane-parallel or spherical clouds, respectively. Readers who wish to see detailed discussions of analogous problems are referred to Davison and Sykes (1957), Chandrasekhar (1960), Case and Zweifel (1967), or Sobolev (1975).

a) Plane-Parallel Geometry

In a plane-parallel medium the transfer equation for the specific intensity $I(r, \mu)$ at spatial coordinate r directed along an angle arccos (μ) with respect to the direction of increasing r is

$$\mu \frac{\partial I(r, \mu)}{\partial r} = -\alpha(r)[I(r, \mu) - S(r, \mu)] , \quad (1)$$

$$S(r, \mu) = \frac{\omega}{4\pi} \int p(\cos \Theta) I(r, \mu') d\Omega' , \quad (2)$$

where $S(r, \mu)$ is the source function for scattered radiation, and the scattering properties are described by the extinction coefficient per unit length $\alpha(r)$, the single scattering albedo $0 \leq \omega \leq 1$ (the fraction of light that is actually absorbed is $1 - \omega$). The phase function for redistribution of scattered light is $p(\cos \Theta)$, where Θ is the angle between the incident and scattered beam.

In terms of the optical-depth scale τ, defined by

$$d\tau = -\alpha(r) dr , \quad (3)$$

the transfer equation assumes the familiar form

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - S(\tau, \mu) . \quad (4)$$

The angular dependence of the specific intensity can be expressed as a series of Legendre polynomials $P_l(\mu)$:

$$I(\tau, \mu) = \sum_{l=0}^{\infty} (2l + 1) F_l(\tau) P_l(\mu) . \quad (5)$$

By the orthogonality properties of Legendre polynomials, it follows the depth-dependent coefficients of this expansion are given by

$$F_l(\tau) = \tfrac{1}{2} \int_{-1}^{+1} I(\tau, \mu) P_l(\mu) d\mu . \quad (6)$$

Substitution of the values $l = 0$, 1, and 2 into this equation yields the useful results

$$F_0 = J ; \quad F_1 = H ; \quad F_2 = 3K - J , \quad (7)$$

where

$$J(\tau) = \tfrac{1}{2} \int_{-1}^{+1} I(\tau, \mu) d\mu = \text{mean intensity} ,$$

$$H(\tau) = \tfrac{1}{2} \int_{-1}^{+1} I(\tau, \mu) \mu d\mu = \text{Eddington flux} ,$$

$$K(\tau) = \tfrac{1}{2} \int_{-1}^{+1} I(\tau, \mu) \mu^2 d\mu = K\text{-moment} . \quad (8)$$

The phase function $p(\cos \Theta)$ can similarly be expanded in Legendre polynomials:

$$p(\cos \Theta) = \sum_{l=0}^{\infty} (2l + 1) \sigma_l P_l(\cos \Theta) . \quad (9)$$

The expansion coefficients σ_l of the phase function can be expressed

$$\sigma_l = \tfrac{1}{2} \int_{-1}^{+1} p(\mu) P_l(\mu) d\mu . \quad (10)$$

The normalization of $p(\mu)$ implies that $\sigma_0 = 1$. The next coefficient σ_1, often denoted by g, represents a mean cosine for the scattering angle:

$$\sigma_1 = g = \tfrac{1}{2} \int_{-1}^{+1} \mu p(\mu) d\mu = \langle \cos \Theta \rangle . \quad (11)$$

For the present paper it will be assumed that the radiation field has azimuthal symmetry about the normal rays. In that case it can be shown (e.g., Chandrasekhar 1960; § 48, eq. [86]) that the source function can be expressed as

$$S(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{+1} R(\mu, \mu') I(\tau, \mu') d\mu' , \quad (12)$$

where

$$R(\mu, \mu') = \sum_{l=0}^{\infty} (2l + 1) \sigma_l P_l(\mu) P_l(\mu') . \quad (13)$$

The expressions for the intensity, equation (5), and phase function, equation (13), are now substituted into the transfer equation (4), using the recurrence relation

$$(2l + 1) \mu P_l(\mu) = (l + 1) P_{l+1}(\mu) + l P_{l-1}(\mu) , \quad (14)$$

satisfied by Legendre polynomials. Equating the coefficients of Legendre polynomials on each side of the equation, we find that the transfer equation assumes the form

$$l F'_{l-1}(\tau) + (l + 1) F'_{l+1}(\tau) = (2l + 1)(1 - \omega \sigma_l) F_l(\tau) , \quad (15)$$

where a prime denotes differentiation with respect to τ.

This is an infinite system of differential equations; a finite approximation to it can be obtained by arbitrarily setting $F_{L+1}(\tau) = 0$ for some odd value of L (the P_L-approximation). The reasons for choosing L odd are discussed by Davison and Sykes (1957) and Case and Zweifel (1967). The approximate system contains $L + 1 = 2M$ equations and $2M$ unknowns. Since the coefficients of the derivatives in equation (15) are constant, solutions for F_l can be found as a sum of $2M$ exponential terms:

$$F_l(\tau) = \sum_m C_{l,m} \exp(-k_m \tau) . \quad (16)$$

Substituting this form into equation (15) and equating coefficients of each exponential yields the relation

$$l C_{l-1,m} k_m - (2l + 1)(1 - \omega \sigma_l) C_{l,m} + (l + 1) C_{l+1,m} k_m = 0 . \quad (17)$$

This is an eigenvalue problem, which can be solved only for certain characteristic values of k_m. In the Appendix it is shown that for $0 \leq \omega < 1$ these characteristic values are real and occur in positive-negative pairs. For convenience the $M = (L + 1)/2$ positive roots are labeled in order of increasing magnitude:

$$0 < k_1 \leq k_2 \leq \cdots \leq k_M . \tag{18}$$

The remaining M roots are labeled such that

$$k_{-m} = -k_m . \tag{19}$$

For each value of m, equation (17) implies certain fixed ratios between the coefficients $C_{l,m}$. It is convenient, therefore, to define $A_m{}^P \equiv C_{0,m}$ and write

$$C_{lm} = A_m{}^P R_{lm} . \tag{20}$$

From equation (17), the ratios R_{lm} satisfy

$$lR_{l-1,m}k_m - (2l + 1)(1 - \omega\sigma_l)R_{l,m} + (l + 1)R_{l+1,m}k_m = 0 . \tag{21}$$

Starting with the values

$$R_{0,m} \equiv 1 , \quad R_{1,m} \equiv (1 - \omega)/k_m , \tag{22}$$

the recurrence relation (21) determines the R_{lm}. It should be emphasized that the R_{lm} do not depend on boundary conditions or the optical thickness of the medium and can be determined once and for all for each choice of ω, σ_l and the order of the approximation L. (The same values of the $R_{l,m}$ also apply to spherical problems. See § II*b*.) Since $k_{-m} = -k_m$, it is easy to show the general result

$$R_{l,-m} = (-1)^l R_{l,m} . \tag{23}$$

Equation (16) can now be written as

$$F_l(\tau) = \sum_{m=-M}^{M}{}' A_m{}^P R_{lm} \exp(-k_m\tau) , \tag{24}$$

where the prime on the summation means to omit the $m = 0$ term. The intensity becomes

$$I(\tau, \mu) = \sum_{l=0}^{L} (2l + 1)P_l(\mu) \sum_{m=-M}^{M}{}' A_m{}^P R_{lm} \times \exp(-k_m\tau) . \tag{25}$$

It now only remains to find the $2M$ constants $A_m{}^P$ by applying the boundary conditions (see § II*c*).

b) *Homogeneous Spherical Geometry*

We next consider the transfer equation for a homogeneous spherical cloud; many aspects of the discussion for plane-parallel clouds will carry over. If the cloud is homogeneous so that $\alpha(r) = \alpha$, and is of total radius R and optical depth τ_c (to the center), then the radius and optical depth to the center are related by

$$\tau_c = \alpha R ,$$

$$\tau_c - \tau = \alpha r , \tag{26}$$

and the transfer equation can be written as

$$\mu \frac{\partial I}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I}{\partial \mu} = -\alpha(I - S) , \tag{27}$$

where the source function remains as in equations (2) and (12).

Again, we seek solutions of the form given in equation (5). We substitute that expression into the spherical transfer equation and eliminate the derivative with respect to μ by the recurrence relation

$$(2l + 1)(1 - \mu^2) \frac{dP_l(\mu)}{d\mu} = l(l + 1)[P_{l-1}(\mu) - P_{l+1}(\mu)] . \tag{28}$$

We find the following relation from the requirement that the coefficient of the lth Legendre polynomial vanishes:

$$l\left(F'_{l-1} - \frac{l-1}{r} F_{l-1}\right) - (2l + 1)(1 - \omega\sigma_l)\alpha F_l + (l + 1)\left(F'_{l+1} + \frac{l+2}{r} F_{l+1}\right) = 0 . \tag{29}$$

The solution to this equation can be written

$$F_l(r) = \sum_{m=1}^{M} C_{lm} i_l(k_m \alpha r) , \tag{30}$$

where i_l is the modified spherical Bessel function of the first kind:

$$i_l(z) = \left(\frac{\pi}{2z}\right)^{1/2} I_{l+1/2}(z) . \tag{31}$$

Here $I_{l+1/2}$ are modified Bessel functions of the first kind. These functions are elementary: for example, $i_0(z) = (\sinh z)/z$. Each term of equation (30) satisfies equation (29) by virtue of the relations (Abramowitz and Stegun 1965; formulae 10.2.20 and 10.2.21)

$$\frac{di_{l-1}(z)}{dz} - \frac{l-1}{z} i_{l-1} = i_l ,$$

$$\frac{di_{l+1}(z)}{dz} + \frac{l+2}{z} i_{l+1} = i_l . \tag{32}$$

Modified spherical Bessel functions of the third kind also obey these relations and also could have been used to construct general solutions; however, being singular at $r = 0$, they were rejected. The resulting equation for the coefficients C_{lm} and k_m is

$$lC_{l-1,m}k_m - (2l + 1)(1 - \omega\sigma_l)C_{lm} + (l + 1)C_{l+1,m}k_m = 0 , \tag{33}$$

which is identical to equation (17) for the plane-parallel case. Thus, we also write

$$C_{lm} = A_m{}^S R_{lm}\,; \quad A_m{}^S \equiv C_{0,m}\,, \tag{34}$$

so that R_{lm} are also identical with those of the plane-parallel case. Now,

$$F_l(r) = \sum_{m=1}^{M} A_m{}^S R_{lm} i_l(k_m \alpha r)\,, \tag{35}$$

and the intensity field is

$$I(r, \mu) = \sum_{l=0}^{\infty} (2l + 1) P_l(\mu) \sum_{m=1}^{M} A_m{}^S R_{lm} i_l(k_m \alpha r)\,. \tag{36}$$

One distinction of the spherical case over the plane-parallel case is that the summation on m is only over positive values, corresponding to positive values of k_m. There are, therefore, only half the number of constants $A_m{}^S$ with which to satisfy boundary conditions. This is correct, because our choice of the nonsingular Bessel functions i_l is already equivalent to having set boundary conditions at $r = 0$, and it only remains to satisfy boundary conditions at the surface. In the plane-parallel case, boundary conditions need to be applied at both surfaces, and this requires twice the number of constants.

c) *Boundary Conditions*

We now formulate boundary conditions appropriate for interstellar clouds surrounded by an ambient radiation field that is isotropic for all angles incident on the cloud's surface. We also assume that the incident radiation is identical on both sides of the plane-parallel clouds and over the entire surface of the spherical clouds. In either geometry the symmetry at the cloud's center unambiguously provides half the required constraints; however, there is no unique prescription for the remaining surface boundary conditions. Here we use *Mark's conditions*, which require the solution to match the incident intensity at M discrete angles μ_i that are the negative roots of the Legendre polynomial of degree $L + 1$. (See Davison and Sykes 1957 for a discussion of Mark's and other surface boundary conditions.)

First consider the plane-parallel case. For a cloud of total optical depth $T = 2\tau_c$, the symmetry condition $I(\tau, \mu) = I(T - \tau, -\mu)$ allows us to eliminate half of the available constants, i.e.,

$$I(\tau, \mu) = \sum_{l=0}^{L} (2l + 1) P_l(\mu) \sum_{m=1}^{M} A_m{}^P R_{lm} \times \{\exp(-k_m \tau) + (-1)^l \exp[k_m(\tau - 2\tau_c)]\}\,. \tag{37}$$

At the surface $\tau = 0$ we can construct a set of M linear equations for the coefficients $A_m{}^P$ by requiring the solution to match the incident intensity I^0 at the M angles μ_i. For each angle μ_i let

$$B_{i,m}{}^P = \sum_{l=0}^{L} (2l + 1) R_{l,m} P_l(\mu_i) \times [1 + (-1)^l \exp(-2k_m \tau_c)]\,. \tag{38}$$

The coefficients $A_m{}^P$ are the solution of a matrix equation of order M:

$$\sum_{m=1}^{M} B_{i,m}{}^P A_m{}^P = I^0\,. \tag{39}$$

Next consider spherical geometry. Application of Mark's condition at $\tau = 0$ leads us to define

$$B_{i,m}{}^s = \sum_{l=0}^{L} (2l + 1) R_{l,m} P(\mu_i) i_l(k_m \alpha R)\,. \tag{40}$$

Then the coefficients $A_m{}^S$ satisfy

$$\sum_{m=1}^{M} B_{i,m}{}^s A_m{}^S = I^0\,. \tag{41}$$

d) *Asymptotic Limit*

Although the general solution involves a sum of M values of k, in the plane-parallel models the factor $\exp(-k\tau)$ in the depth dependence guarantees that, for clouds of sufficient optical depth, the asymptotic solution at great depth will be dominated by the smallest characteristic value k_1. The growing mode $\exp(k_1\tau)$ in the plane-parallel solution represents the influence of radiation incident on the far boundary and is of negligible importance until, at the cloud's center, it contributes half the mean intensity. Since the functions $i_l(z)$ are asymptotically

$$i_l(z) \sim \frac{1}{2z} e^z\,, \quad z \to \infty\,, \tag{42}$$

the spherical solutions are also dominated by the lowest characteristic value k_1.

The explicit form of the mean intensity in the asymptotic limit is

$$J(\tau) = I^0 A_1{}^P \{\exp(-k_1\tau) + \exp[k_1(\tau - 2\tau_c)]\} \tag{43}$$

in the plane-parallel case and

$$J(\tau) = I^0 A_1{}^s \frac{\sinh[k_1(\tau_c - \tau)]}{k_1(\tau_c - \tau)} \tag{44}$$

in the spherical case.

In order to know when the asymptotic limit applies, we must also find the next larger root k_2. Then, barring large differences in the boundary coefficients, the eigenvector associated with eigenvalue k_1 will dominate the solution for optical depths greater than τ_A when

$$\exp[(k_2 - k_1)\tau_A] \gg 1\,. \tag{45}$$

III. CLOUD MODELS WITH A HENYEY-GREENSTEIN PHASE FUNCTION

We apply the techniques developed above to generate models of interstellar clouds with a Henyey-Greenstein redistribution function. The models depend on three parameters: (1) τ_c the optical depth to the cloud's center, (2) ω the single scattering albedo, and (3) g the mean cosine of the scattering angle. If Θ is the angle between incident and scattered beams, then

$$p(\cos \Theta) = \frac{1 - g^2}{[1 + g^2 - 2g \cos \Theta]^{3/2}} \tag{46}$$

is the one-parameter phase function proposed by Henyey and Greenstein (1941) which, by design, takes a particularly simple form as an expansion in Legendre polynomials:

$$p(\cos \Theta) = \sum_{l=0}^{\infty} (2l + 1) g^l P_l(\cos \Theta) , \tag{47}$$

so that the σ_l have the values

$$\sigma_l = g^l . \tag{48}$$

From equation (11) we see that g indeed has its previous meaning as the mean cosine of the scattering angle. As g varies from 0 to 1, the scattering changes from isotropic to completely forward scattering. Obviously, the latter case is equivalent to pure absorption (that is, no scattering integral appears), with a new optical-depth scale $\tau' = (1 - \omega)\tau$. This degeneracy of the scattering problem with an equivalent problem involving pure absorption will explain several properties of the solutions found below as $g \to 1$.

An analytic treatment similar to ours is given in Sobolev (1975) for a semi-infinite plane-parallel atmosphere. In particular he discusses the characteristic equation and the existence of an asymptotic solution for the decay of mean intensity, and he gives values for that decay rate as a function of ω, g. (See also van de Hulst 1970). We extend these discussions to include the complete solution for both plane-parallel and spherical models, with isotropic incident fields. Our solutions for spherical models are compared with those of Sandell and Mattila (1975), which were derived by Monte Carlo simulations.

a) Roots of the Characteristic Equation

To apply the preceding theory we need the characteristic roots k_m. Analytic expressions for the characteristic equation can be obtained for either isotropic, $g = 0$, or pure forward scattering, $g = 1$:

$$\tanh^{-1}(k)/k = 1/\omega , \qquad g = 0 ; \tag{49a}$$

$$k = 1 - \omega , \quad g = 1 . \tag{49b}$$

The result for $g = 0$ follows by comparing equation (A6) with the continued fraction representation for $\tanh^{-1}(x)/x$; it is valid for all k. The result for forward scattering follows by noting that when $g = 1$, equation (A6) takes a form identical to equation (49a) in the special case $\omega = 0$, $g = 0$, if we replace k^2 by $k^2/(1 - \omega)$. This special case of equation (49a) has the limiting solution $k^2 \to 1$ as $\omega \to 0$, which demonstrates the particular solution for $g = 1$. For intermediate values of g, $0 < g < 1$, equation (A6) must be solved numerically (see Figs. 1 and 2).

There are two regimes where the roots are not well separated; both correspond to situations where scattering is negligible compared with pure absorption. The first regime occurs when the smallest root is

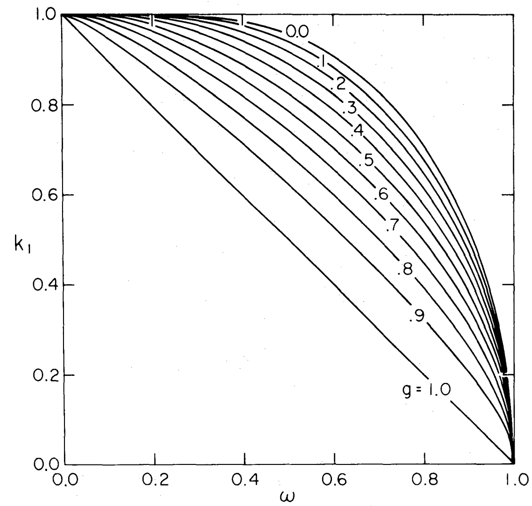

FIG. 1.—The characteristic value k_1 as a function of the single scattering albedo ω for various values of the phase factor g in the Henyey-Greenstein phase function. The roots were found by numerical solution of the characteristic equation for a sufficiently large order that the root had converged to at least three significant figures.

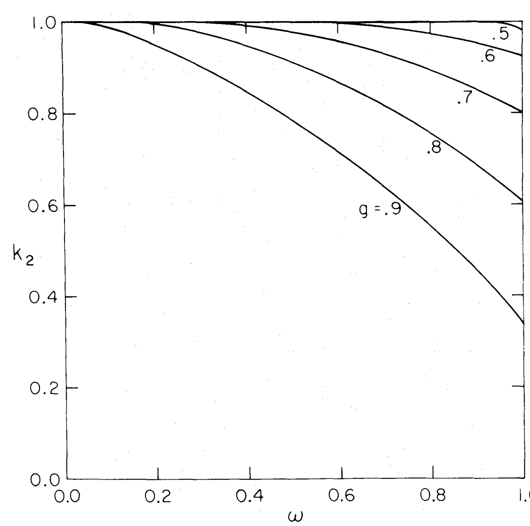

FIG. 2.—The second characteristic value as a function of ω, g (see Fig. 1). Where not shown, the second root becomes arbitrarily close to unity (see text).

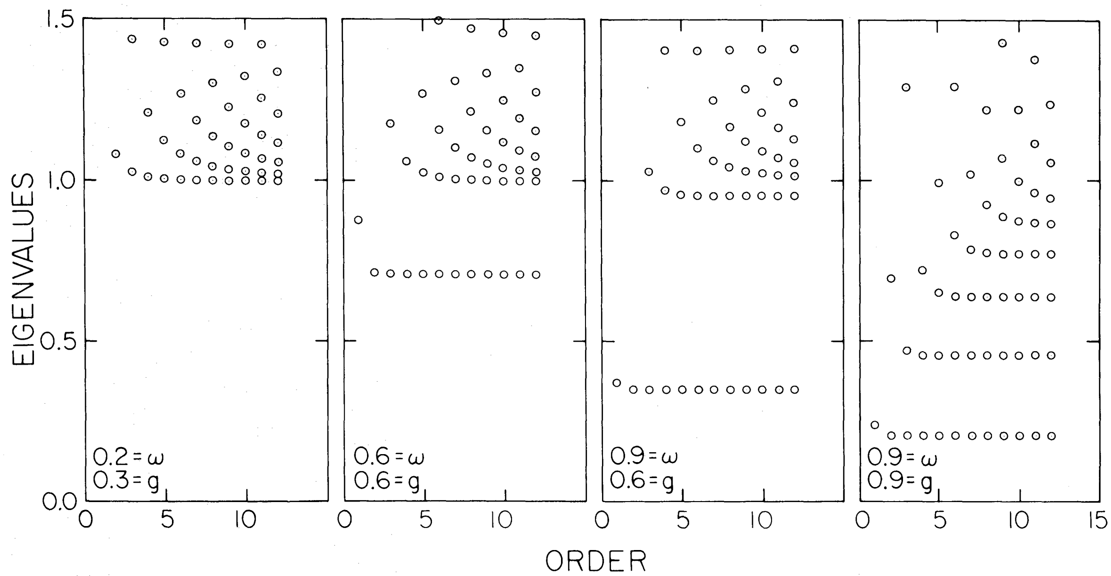

FIG. 3.—Characteristic values as a function of the order of the polynomial approximation. Shown are the positive roots $k_m < 1.5$ as a function of the order $M = (L + 1)/2$ of the characteristic polynomial in k^2, where L is the highest order Legendre polynomial in the approximate solution. Note that a low-order approximation suffices to represent k_1 with high accuracy.

near unity, where, obviously, pure absorption dominates. For pure absorption the mean intensity at a point can be evaluated as an integral over angle of the incident intensity exponentially attenuated according to the path length; approximated as a sum of terms $\exp(-k\tau)$, the sum requires a continuous range of decay rates $k > 1$ to cope with the variable path lengths. The second regime in which the low order roots coalesce occurs when $g \to 1$. As mentioned above, this corresponds to a pure absorption problem with a new optical-depth scale $\tau' = (1 - \omega)\tau$. When the depth scale is not altered, the continuum of roots begins at the minimum root $k_1 = (1 - \omega)$.

In Figure 3 we display the behavior of the eigenvalues, k_m, as the order of the approximation increases. For $L = 1$, we recover the so-called "diffusion approximation"

$$k_1^2 = 3(1 - \omega)(1 - \omega g) \tag{50}$$

for the smallest (and only root). However, as the order of approximation increases, the smallest root k_1 always decreases. The diffusion approximation always overestimates the asymptotic rate of decay of the mean intensity. Three points should be noted from the figure. First, only a low order approximation is necessary to obtain an accurate value for the decay rate of the asymptotic solution; in most cases $M = 2$ suffices. Second, regardless of the size of k_1, k_2 never exceeds 1 by more than an arbitrarily small amount for a sufficiently high order approximation. From the discussion of the preceding paragraph this follows because a continuum of decay rates, starting from $k = 1$, are required to represent the direct decay of the absorbed incident radiation. Third, for $g = 0.9$ the figure shows that many roots with $k < 1$ begin to appear, which suggests the need for a continuum of roots above $1 - \omega$ as $g \to 1$; in fact, numerical solutions of the equation become difficult in this regime.

For those wishing to utilize this method, the following properties of the characteristic equation will assist in its solution. A lower bound on the smallest root is provided by equation (49b). For approximations of high order, extended precision is necessary to evaluate the continued fraction or characteristic polynomial, and the higher roots may be closely spaced. In proceeding from order n to $n + 1$ the roots of the lower order approximation interleave between the roots of the higher order equation. Algorithms exist (see, e.g., Abramowitz and Stegun 1965, § 3.10.11) that allow one to extend the computation of a continued fraction to the next higher level without restarting.

b) *Cloud Models*

Figure 4 displays the mean intensity of radiation, normalized relative to the incident specific intensity, as a function of optical depth for a series of models with plane-parallel (*top row*) and spherical (*bottom row*) geometry. In the figure for each pair ω, g we show models with central optical depth $\tau_c = 5$, 10, 15, 20 to illustrate the behavior of the depth dependence. Although the models contain a complete description of the angular dependence of the radiation field (within the accuracy of the order of approximation, $M = 11$ for all models), physically it is the first moment of the intensity, the mean intensity J, which enters all radiative rates. To obtain only the first moment from the double-sum series representation of the solution, equation (36) or (37), requires a

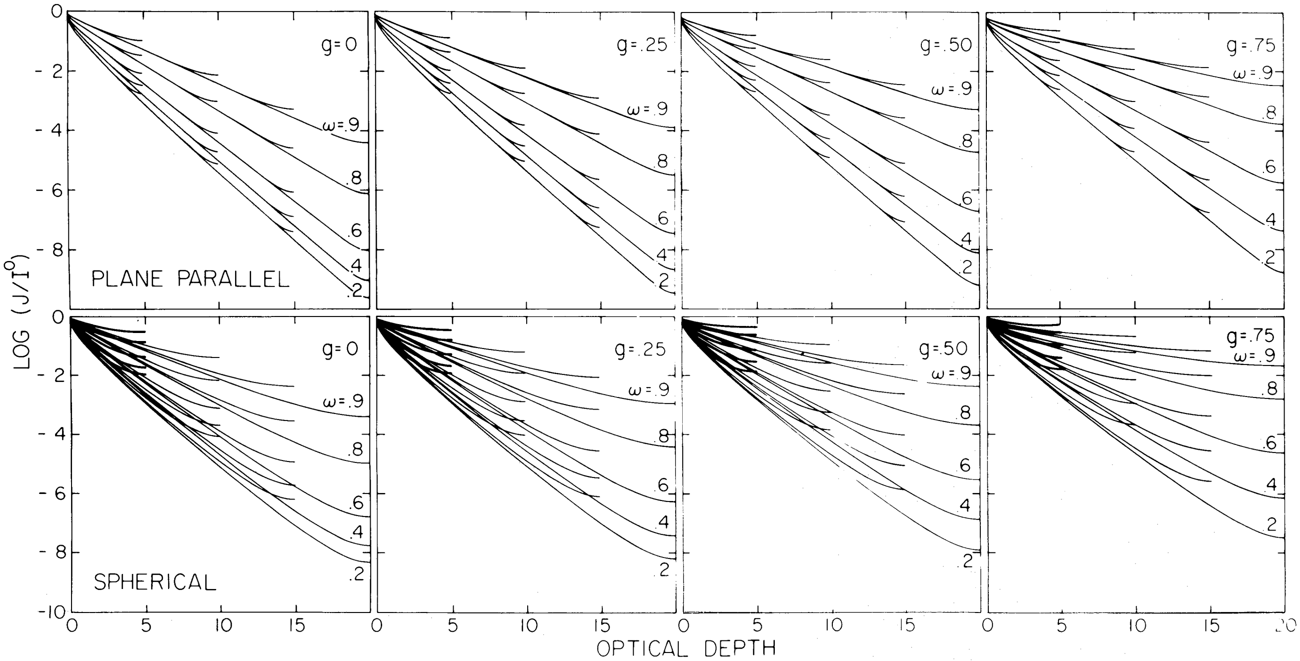

FIG. 4.—Mean intensity versus optical depth. The models depend on three parameters: ω the single scattering albedo, g the phase factor in the Henyey-Greenstein phase function, and τ_c the optical depth to the center of the cloud. For various combinations of ω, g the curves show the depth-dependent solution for the mean intensity, normalized relative to the incident specific intensity I^0, for a series of models with optical depth to the cloud center $\tau_c = 5, 10, 15, 20$. Models in the top (bottom) row are for clouds with plane-parallel (spherical) geometry, and were calculated with truncation at order $M = 11$ in the Legendre expansion. Note that most models show a characteristic asymptotic decay at large optical depth.

summation over only the index m with $l = 0$, and $R_{0,m} = 1$. Qualitatively the solutions show the behavior pointed out by Sandell and Mattila (1975): for increasing albedo or anisotropy of scattering the radiation penetrates more effectively.

In most cases the solution achieves its asymptotic form by the time $\tau > 5$. Note that in all models the solution exhibits a gentle rise near the center in response to the effects of radiation from the other boundaries. Near the surface the terms associated with larger characteristic values contribute substantially to the mean intensity.

Figure 4 and the form of the asymptotic solution show the strong dependence of the mean intensity on the geometry of the cloud. The difference appears explicitly in the additional $1/[k(\tau_c - \tau)]$ factor in the spherical asymptotic solution. In all cases at similar optical depth the mean intensity is greater in spherical clouds than in plane-parallel clouds of the same total optical depth. Obviously, at any point and for any angle (except radial), the path length to the boundary in a plane-parallel cloud exceeds that length in a spherical cloud.

Our solutions for spherical clouds differ in two important ways from similar models by Sandell and Mattila (1975). First, we find substantial quantitative disagreement, often by factors exceeding an order of magnitude, with their results for the central mean intensity; in all cases their results are too large. Probably this discrepancy arises from their definition of "core" intensity. In their Monte Carlo simulation they assign to the core all photons which penetrate to within 10% of the cloud's center by radius. For clouds of large optical depth, 10% by radius still represents a substantial depth, e.g., $\Delta\tau = 2$, for $\tau_c = 20$. They do not indicate that any correction was applied to account for the remaining attenuation, so this effect could explain some or all of the differences between our results for the central mean intensity.

The second difference concerns the depth-dependent solutions. Sandell and Mattila suggest that the solution for mean intensity with optical depth, in a cloud of arbitrary central optical depth, can be obtained by interpolation in their solutions for central mean intensity with central optical depth. This is equivalent to constructing a universal intensity versus optical-depth curve (for each pair ω, g) from the solutions shown in Figure 4 by joining the points representing the central mean intensity for clouds of increasing central depth. This manifestly overestimates the mean intensity. Deep in optically thick clouds the overestimate is a factor of 2 at least, since the central mean intensity deviates from the asymptotic solution by that factor in response to the influence of the other boundaries. At small optical depths in thick clouds the error can exceed a factor of 2 as is apparent from the figure.

We also point out two minor differences in the form of our results and those of Sandell and Mattila. First, they plot results for the diametrical optical depth, while we use the optical depth to center, one-half their value. Second, we normalize the mean intensity relative to the incident specific intensity, which we take to be isotropic in the half-space covering incident angles. Since the emergent intensity cannot exceed the incident one, our solutions at $\tau = 0$ always satisfy $J(0)/I^0 < 1$. However, Sandell and Mattila chose some normalization such that all their plotted solutions start from $I_c/I_0 = 1$ (in their notation) at $\tau = 0$. This normalization already implies that their solutions exceed ours, but it does not explain all of the discrepancies.

c) *Asymptotic Decay Rates in Interstellar Clouds*

Before our mathematical formalism can be applied to model interstellar clouds we must first know the scattering properties of real grains. In another paper (Roberge, Dalgarno, and Flannery 1980) we apply these mathematical methods to derive radiative lifetimes for several atoms and molecules of astrophysical interest and present a critical discussion of grain properties. In clouds of sufficient optical depth the asymptotic solution controls the UV spectrum at great depth. Here, we apply our results for the asymptotic solution of the transfer equation to derive the relative attenuation of incident radiation as a function of wavelength using accepted values for the grain properties.

A conventional representation for the grain properties can be constructed from the extinction curve of Bless and Savage (1972) and the grain-scattering properties ω, g found by Lillie and Witt (1976) by modeling observations of diffuse galactic light. For a cloud of sufficient optical depth we can represent the relative decay of mean intensity with visual optical depth $k_V(\lambda)$ as the product of the smallest eigenvalue as a function of ω, g times the attenuation relative to visual:

$$k_V(\lambda) = k_1(\omega, g)\sigma_\lambda/\sigma_V \,. \tag{51}$$

Our results for the relative attenuation are shown in Figures 5 and 6. A major uncertainty concerns the behavior of the scattering properties for the short wavelengths, 912 Å < λ < 1500 Å, so we demonstrate the effect of three hypothetical extensions of the grain properties to short wavelengths. The important point is that, although the extinction curve rises sharply to shorter wavelengths, the actual attenuation of incident radiation may be less as a result of the offsetting increase in the scattered radiation. This result has, of course, been pointed out before, notably by Sandell and Mattila, but our result takes a particularly simple analytic form. The uncertainty at short wavelengths has particular importance for the rates associated with the abundance of the critical constituents C^+ and H_2.

IV. THE WHITWORTH PHASE FUNCTION

Whitworth (1975) proposed a model for dust scattering in which a certain fraction $(1 - g)$ of the scattered radiation was redistributed isotropically, and the remaining fraction g was "scattered" into the forward direction, $\Theta = 0$. The phase function thus has the form

$$p(\cos\Theta) = (1 - g) + 2g\delta(\cos\Theta - 1)\,. \tag{52}$$

An expansion in Legendre polynomials yields an equation of the form equation (9), where

$$\sigma_0 = 1\,,$$
$$\sigma_l = g\,, \quad l \geq 1\,. \tag{53}$$

The $\sigma_0 = 1$ value follows from the normalization of the phase function. Since $\sigma_1 = g$, the quantity g again has the meaning of a mean cosine of the scattering angle.

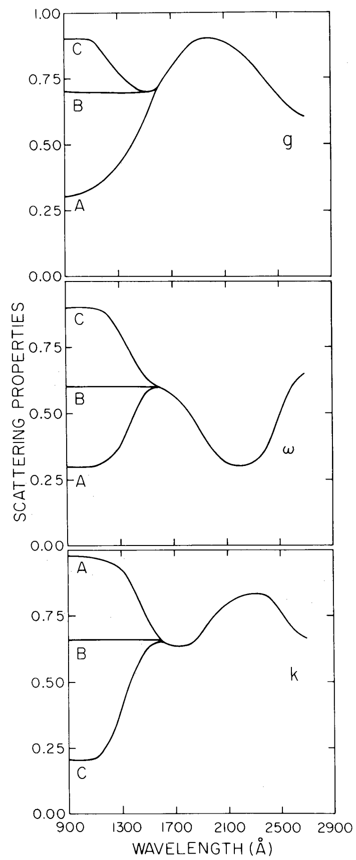

FIG. 5.—Grain parameters ω, g after Lillie and Witt (1976) with associated smallest characteristic values k_1. Parameter values for ω, g for $\lambda <$ 1500 Å represent three hypothetical extrapolations. The characteristic value k_1 gives the exponential decay rate of mean intensity with optical depth in the asymptotic limit.

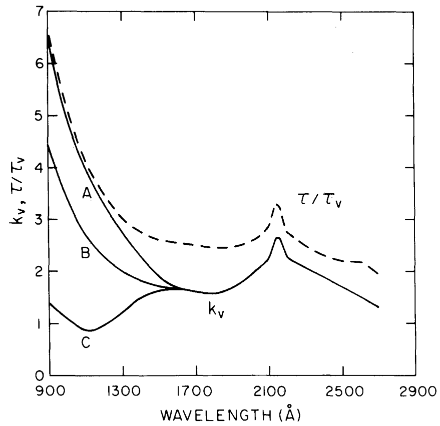

FIG. 6.—The relative extinction curve and asymptotic decay rates, $J/I^0(\lambda) \propto \exp(-k_V(\lambda)\tau_V)$, versus wavelength. Values for $\lambda < 1500$ Å are derived from hypothetical extrapolations of the Lillie and Witt ω, g parameters as in Fig. 5. The extinction curve of Bless and Savage (1972) is used to convert the asymptotic decay rates shown in Fig. 5 to a common (visual) optical-depth scale. In spite of the sharp increase of extinction at shorter wavelengths, when scattering effects are considered, it is possible for the mean intensity to decay more gradually with decreasing wavelength.

The transfer equation for the Whitworth model is, from equations (2) and (4),

$$\mu \frac{\partial I}{\partial \tau} = I - (1-g)\omega J - \omega g I\,,$$
$$= (1-\omega g)\left(I - \frac{(1-g)\omega}{1-\omega g} J\right)\,. \tag{54}$$

This can be written

$$\mu \frac{\partial I}{\partial \tau_e} = I - \omega_e J\,, \tag{55}$$

where the "effective" optical depth and "effective" albedo are defined by

$$\tau_e = (1-\omega g)\tau\,,$$
$$\omega_e = \frac{\omega(1-g)}{1-\omega g}\,. \tag{56}$$

Thus the Whitworth model is equivalent to an isotropic scattering problem in rescaled variables τ_e and ω_e.

Solutions using the Whitworth phase function can be found as an appropriate rescaled case of isotropic scattering (or the Henyey-Greenstein function for $g = 0$). In particular, for given values of parameters τ_c, g, and ω the Whitworth solution at optical depth τ is the same as the isotropic solution for the parameters $\tau_{ce} = (1-\omega g)\tau_c$, $g = 0$, and $\omega_e = (1-g)\omega/(1-\omega g)$ at optical depth $\tau_e = (1-\omega g)\tau$.

Whitworth advanced the phase function in equation (52) as a simplified model for the more complicated Henyey-Greenstein function. Equation (53) shows that Legendre expansions of the two functions agree through the first two terms, but then deviate for higher coefficients—where Henyey-Greenstein has $\sigma_l = g^l$ (not g). To determine the similarity in consequences between the two phase functions, we undertook a comparison of detailed models. We generated a sequence of models with the Whitworth phase function by using the analogous isotropic case described in the preceding paragraph and compared the mean intensity of each model relative to its Henyey-Greenstein counterpart.

Models with small optical depth agree quite well, but as the optical depth increases, the agreement progressively deteriorates. That behavior arises from small differences between the decay rates of the two models. The asymptotic decay rates differ by at most 15%, the largest discrepancies occur for moderate values of ω and for values of g close to unity. So, for small optical depths the effect on mean intensity can be small, but any difference in the decay rate, however small, will lead to large relative differences for sufficiently large optical depths.

V. DISCUSSION

The techniques presented here allow one efficiently and accurately to calculate the depth-dependent radiation field in interstellar clouds. The procedure is mostly analytic, involving no numerical sophistication beyond solving a characteristic equation with real, positive roots and solving a real matrix equation to derive the weighting coefficients from the boundary conditions. The accuracy of solutions can be assessed simply by extending the expansion to include more terms. In many cases satisfactory solutions can be obtained after truncation of the Legendre expansion at third order ($L = 3$), which only requires solution of a quadratic polynomial and inversion of a 2×2 matrix.

We have compared solutions for the mean intensity in plane-parallel models obtained from the finite Legendre polynomial expansion against solutions derived by standard N stream techniques. For the same number of terms, discrete angles or Legendre polynomials, the Legendre expansion is more accurate. This is not surprising since the isotropic incident field already favors low order representations in Legendre expansions. Since the Legendre technique readily handles both plane-parallel and spherical clouds, it is always to be preferred.

One important extension of the present methods would be to cases where there are sources of photons internal to the medium, as well as the photons incident on the boundaries. Thus equations (1) and (2) are replaced by

$$\mu \frac{\partial \bar{I}(r,\mu)}{\partial r} = -\alpha(r)[\bar{I}(r,\mu) - \bar{S}(r,\mu)]\,, \tag{57}$$

$$\bar{S}(r,\mu) = \frac{\omega}{4\pi}\int p(\cos\Theta)\bar{I}(r,\mu')d\Omega' + S^*(r)\,, \tag{58}$$

where $S^*(r)$ defines the strength of the internal sources.

We remark that for the particular case of homogeneous sources, $S^*(r) = S^*$, with no incident radiation, it is possible to give the solution in terms of the case treated in this paper. Setting $B = S^*/(1 - \omega)$, the solution can be written

$$\bar{I}(r, \mu) = B - I(r, \mu)\,, \tag{59}$$

where $I(r, \mu)$ satisfies equations (1) and (2) with $I^0 = B$, and can be found from the formulae of § II.

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APPENDIX

PROPERTIES OF THE CHARACTERISTIC VALUES

In the P_L-approximation one can display equation (21) using a symmetric tridiagonal matrix D (we omit the index m):

$$\begin{bmatrix} -h_0 & k & & & & \\ k & -h_1 & 2k & & & \\ & 2k & -h_2 & 3k & & \\ & & 3k & -h_3 & & \\ & & & & \vdots\vdots\vdots & Lk \\ & & & & Lk & -h_L \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_L \end{bmatrix} = 0\,. \tag{A1}$$

The diagonal elements are

$$h_l \equiv (2l + 1)(1 - \omega\sigma_l)\,. \tag{A2}$$

From equation (10), the normalization of $p(\mu)$ and the fact that $|P_l(\mu)| \leq 1$ for $|\mu| \leq 1$, it follows that $|\sigma_l| \leq 1$ for all values of l. Thus the diagonal elements h_l and their square roots $h_l^{1/2}$ are positive and real for $0 \leq \omega < 1$. If we incorporate into each element of the vector R_l the factor $h_l^{1/2}$ and divide each row of the matrix by $kh_l^{1/2}$, then the resulting matrix equation

$$\begin{bmatrix} -k^{-1} & (h_0h_1)^{-1/2} & \\ (h_0h_1)^{-1/2} & -k^{-1} & 2(h_1h_2)^{-1/2} \\ & 2(h_1h_2)^{-1/2} & -k^{-1} \\ & & \vdots\vdots\vdots \end{bmatrix} \begin{bmatrix} h_0^{1/2}R_0 \\ h_1^{1/2}R_1 \\ h_2^{1/2}R_2 \\ \vdots \end{bmatrix} = 0\,, \tag{A3}$$

matches an eigenvalue problem for a real, symmetric matrix, with eigenvalues that are inverses of the characteristic values k. Thus the $2M$ solutions for k must be real.

To show that the roots occur in positive-negative pairs, note that if equation (A1) is to have a nonvanishing solution for the coefficients R_l, then the determinant of the matrix must be zero $F_l(k) \equiv \det |D_l| = 0$. This condition on k is called the *characteristic equation*. The determinant of the order $2M$ tridiagonal matrix can readily be expanded by minors to obtain a recurrence relation in terms of the lower order determinants:

$$F_0(k) = -h_0\,, \tag{A4a}$$

$$F_1(k) = h_1h_2 - k^2\,, \tag{A4b}$$

$$F_l(k) = h_lF_{l-1}(k) - l^2k^2F_{l-2}(k)\,. \tag{A4c}$$

It is apparent that the dependence of the characteristic equation on k enters only through the factor k^2. Thus, the $2M$ roots are real and occur in pairs $\pm k_m$, $m = 1, 2, \ldots, M$.

The characteristic equation can also be expressed in terms of a continued fraction. With the definition $\rho_l = lk_m(R_{lm}/R_{l-1,m})$ and equation (21) we have

$$\rho_l = \frac{l^2k^2}{h_l - \rho_{l+1}}\,. \tag{A5}$$

From equation (22) it follows that $\rho_1 = 1 - \omega$. By using equation (A5) repeatedly, we find the representation for the characteristic equation (van de Hulst 1970; Sobolev 1975)

$$1 - \omega = \frac{k^2}{h_1^-}\,\frac{4k^2}{h_2^-}\,\frac{9k^2}{h_3^-}\cdots . \tag{A6}$$

In the P_L approximation, this continued fraction terminates after L terms. Other forms for the characteristic equation are given in van de Hulst (1970).

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