Formal RT solvers

- Long characteristics (Feautrier scheme)
  Cannon C.J. 1970, *Astrophys. J.*, v.161, p.255
- Short characteristics (Attenuation operator)
  Bruls et al. 1999, *Astron. and Astroph.*, v.348, p.233
- Short characteristics (Hermitian scheme)
  Bellot Rubio et al. 1998, *Astrophys. J.*, v.506, p.805

Why formal solver?

- Once we found the source function we need find the intensities in order to compute the energy transport
- The intensities are needed to compute self-consistent level populations $\rightarrow$ opacities $\rightarrow$ optical depth scale
- To get a consistent solution we may need to iterate

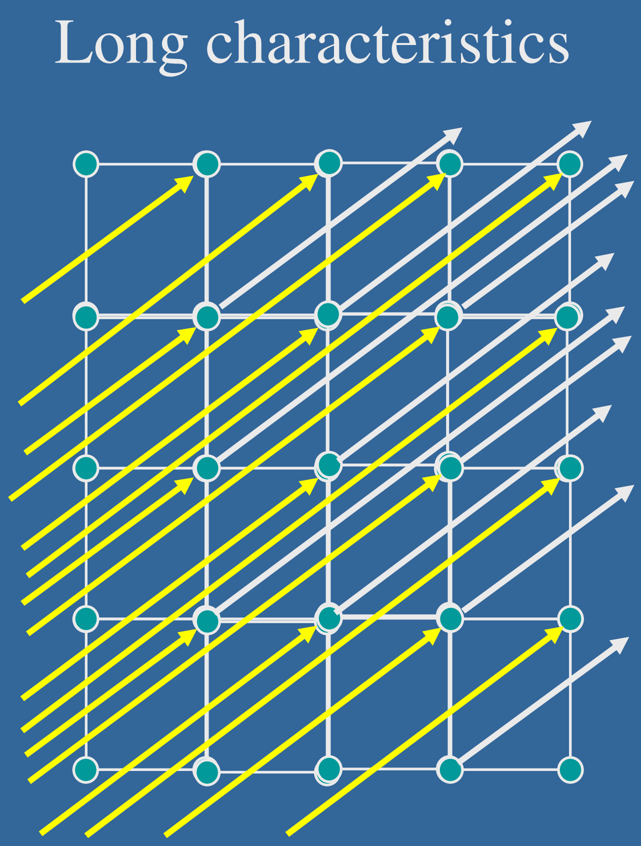
Requirements for a formal solver

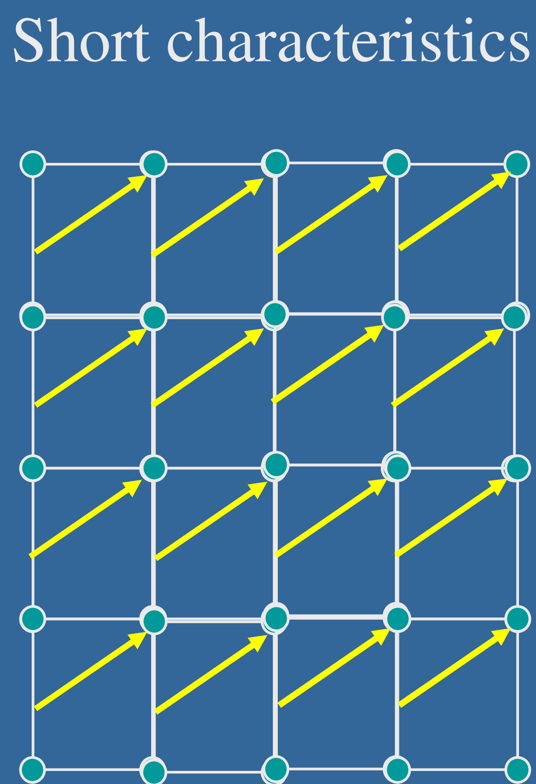
- The solver for intensities must be very quick and stable. It should be able to achieve good accuracy in the prescribed grid on which the source function is known
- The method should not propagate/amplify errors which may occur in early iterations
- The last two requirements make impossible to use RK in NLTE and significant scattering

Method classification

- RT solvers act along rays or *characteristics* that do not necessarily coincide with the selected grid.
- Individual ray can be followed through the whole medium boundary-to-boundary or over a short part extending the length of one grid cell.
- RT solvers based on complete rays are known as *long characteristics* methods.
- RT solvers that follow radiation through a single cell at a time are called *short characteristics* methods.
- In 1D there is obviously no difference between short and long characteristic methods

Method classification





Feautrier RT solver

- Equation of radiative transfer (again)

$$\frac{dI_\nu}{dx} = -k_\nu \rho \cdot (I_\nu - S_\nu)$$

where x is a geometrical distance along the ray

- Let's split the intensity in two flows: I^+ in the direction of increasing x and I^- in the opposite direction. The RT equation can be written for each direction:

$$\frac{dI_\nu^+}{dx} = -k_\nu \rho \cdot (I_\nu^+ - S_\nu)$$

$$-\frac{dI_\nu^-}{dx} = -k_\nu \rho \cdot (I_\nu^- - S_\nu)$$

We define two new variables $U = \frac{1}{2}(I^+ + I^-)$ and $V = \frac{1}{2}(I^+ - I^-)$. Now we can add/subtract the two equations of RT and divide the results by 2:

$$\frac{dI_\nu^+}{dx} = -k_\nu \rho \cdot (I_\nu^+ - S_\nu)$$

$$+$$

$$\frac{dI_\nu^-}{dx} = \quad k_\nu \rho \cdot (I_\nu^- - S_\nu)$$

$$\frac{dU_\nu}{dx} = -k_\nu \rho \cdot V_\nu$$

$$\frac{dI_\nu^+}{dx} = -k_\nu \rho \cdot (I_\nu^+ - S_\nu)$$

$$-$$

$$\frac{dI_\nu^-}{dx} = \quad k_\nu \rho \cdot (I_\nu^- - S_\nu)$$

$$\frac{dV_\nu}{dx} = -k_\nu \rho \cdot (U_\nu - S_\nu)$$

2nd order form of RT

- We substitute the derivative of V in the 2nd equation using the expression for V from the 1st equation:

$$V_\nu = -\left(k_\nu \rho\right)^{-1} \frac{dU_\nu}{dx}$$

$$\frac{dV_\nu}{dx} = -k_\nu \rho \cdot (U_\nu - S_\nu)$$

- The equations for U and V can be combined to a single 2nd order ODE:

$$\frac{d}{dx}\left[\left(k_\nu \rho\right)^{-1} \frac{dU_\nu}{dx}\right] = k_\nu \rho \cdot (U_\nu - S_\nu)$$

Boundary Conditions

Boundary conditions are set in the two ends of the medium. For the smallest x we can write:

$$\left(k_\nu \rho\right)^{-1} \left.\frac{dU_\nu}{dx}\right|_A = -V_\nu = -\frac{1}{2}\left(I_\nu^+ - I_\nu^-\right) =$$

$$= \frac{1}{2}\left(I_\nu^+ + I_\nu^-\right) - I_\nu^+ = U_\nu - I_\nu^A$$

For the opposite end we have:

$$\left(k_\nu \rho\right)^{-1} \left.\frac{dU_\nu}{dx}\right|_B = -V_\nu = -\frac{1}{2}\left(I_\nu^+ - I_\nu^-\right) =$$

$$= -\frac{1}{2}\left(I_\nu^+ + I_\nu^-\right) + I_\nu^- = -U_\nu + I_\nu^B$$

Finite differences equation have familiar form (*note the sign change*):

$$-a_i U_{i-1} + b_i U_i - c_i U_{i+1} = d_i \quad \text{for} \quad i = 2, \ldots, N-1$$

$$a_i = \frac{1}{x_{i+1} - x_{i-1}} \cdot \frac{\left(k_{\nu,i}\rho\right)^{-1} + \left(k_{\nu,i-1}\rho\right)^{-1}}{x_i - x_{i-1}}$$

$$c_i = \frac{1}{x_{i+1} - x_{i-1}} \cdot \frac{\left(k_{\nu,i+1}\rho\right)^{-1} + \left(k_{\nu,i}\rho\right)^{-1}}{x_{i+1} - x_i}$$

$$b_i = a_i + c_i + k_{\nu,i}\rho$$

$$d_i = k_{\nu,i}\rho \cdot S_{\nu,i}$$

For i=1 we can write linear boundary condition:

$$\frac{U_2-U_1}{\tau_2-\tau_1} \approx \left.\frac{dU}{d\tau}\right|_{\tau_{1\frac{1}{2}}} \quad !$$

$$U_{1\frac{1}{2}} = \frac{U_2-U_1}{2} \approx U_1 + \frac{(\tau_2-\tau_1)}{2}\cdot \left.\frac{dU}{d\tau}\right|_1 =$$

$$= U_1 + \frac{(\tau_2-\tau_1)}{2}\cdot(U_1+I^A)$$

$$\frac{(\tau_2-\tau_1)}{2} \approx \frac{(k_2\rho+k_1\rho)}{2}\cdot\frac{x_2-x_1}{2}$$

$$U_1\cdot\underbrace{\left[1+\frac{(k_2\rho+k_1\rho)}{2}\cdot(x_2-x_1)\right]}_{b_1} - \overbrace{1}^{c_1}\cdot U_2 = \underbrace{\frac{(k_2\rho+k_1\rho)}{2}\cdot(x_2-x_1)}_{d_1}\cdot I^A$$

… or we can write quadratic boundary condition:

$$U_2 = U_1 + \delta\tau \left.\frac{dU}{d\tau}\right|_{\tau_1} + \frac{1}{2}\delta\tau^2 \left.\frac{d^2U}{d\tau^2}\right|_{\tau_1} + \ldots \approx$$

$$\approx U_1 + \delta\tau \cdot (U_1 - I_\nu^A) + \frac{1}{2}\delta\tau^2 \cdot (U_1 - S_1)$$

$$\delta\tau \approx \frac{(k_2\rho + k_1\rho)}{2} \cdot (x_2 - x_1)$$

$$U_1 \cdot \underbrace{\left[1 + \delta\tau + \frac{1}{2}\delta\tau^2\right]}_{b_1} - \overbrace{1}^{c_1} \cdot U_2 =$$

$$= \underbrace{\delta\tau \cdot I^A + \frac{1}{2}\delta\tau^2 S_1}_{d_1}$$

The case of i=N is similar

For semi-infinite medium boundary condition at ∞ looks a bit different:

$$\begin{cases} \dfrac{dU_\nu}{d\tau} = U_\nu - I_\nu^+ \\ \begin{cases} \dfrac{dI_\nu^+}{d\tau} = S_\nu - I_\nu^+ \\ \lim\limits_{x\to\infty} I_\nu^+ = S_\nu \end{cases} \Rightarrow I_\nu^+\Big|_{\text{deep}} = S_\nu + \dfrac{dS_\nu}{d\tau} \end{cases} \Rightarrow U_\nu - \frac{dU_\nu}{d\tau} = S_\nu + \frac{dS_\nu}{d\tau}$$

$$c_1 = \frac{1}{\delta\tau} - \frac{1}{2}$$

$$b_1 = \frac{1}{\delta\tau} + \frac{1}{2}$$

$$d_1 = \tfrac{1}{2}(S_{\nu,2} + S_{\nu,1}) + \frac{(S_{\nu,2} - S_{\nu,1})}{\delta\tau}$$

Attenuation operator solver

- Solution of RT over one grid cell can be written:

$$I_\nu(\tau_{i+1}) = e^{-(\tau_{i+1}-\tau_i)} \cdot I_\nu(\tau_i) +$$

$$+ \int_{\tau_i}^{\tau_{i+1}} S_\nu(t) \cdot e^{-(\tau_{i+1}-t)} dt$$

where τ is the optical path along the ray

- Suppose S slowly changes with τ which can be approximated by a linear function. Then we can take the integral analytically!

$$S_\nu(\tau) = \left[\frac{(\tau_{i+1}-\tau)}{(\tau_{i+1}-\tau_i)} S_{\nu,i} + \frac{(\tau-\tau_i)}{(\tau_{i+1}-\tau_i)} S_{\nu,i+1} \right]$$

$$I_\nu(\tau_{i+1}) = I_\nu(\tau_i) \cdot e^{-(\tau_{i+1}-\tau_i)} + \eta_{\nu,i}$$

How this works?

- Select direction
- For starting grid points incoming intensity is given by boundary conditions
- To compute the opacity and the source function in the starting and the following points we may need to interpolate. These are needed to compute the intensity in the next point
- One can take 3 points and make parabolic fit to the source function

Hermitian method

Taylor expansion for intensity in point τ_i:

$$I_{i+1} = I_i + \sum_{n=1}^{4} \frac{\delta_i^n}{n!} \frac{d^n I}{d\tau^n}$$

$$I' = \frac{dI}{d\tau} + \delta_i \frac{d^2 I}{d\tau^2} + \frac{1}{2}\delta_i^2 \frac{d^3 I}{d\tau^3} + \frac{1}{6}\delta_i^3 \frac{d^4 I}{d\tau^4}$$

$$I'' = \frac{d^2 I}{d\tau^2} + \delta_i \frac{d^3 I}{d\tau^3} + \frac{1}{2}\delta_i^2 \frac{d^4 I}{d\tau^4}$$

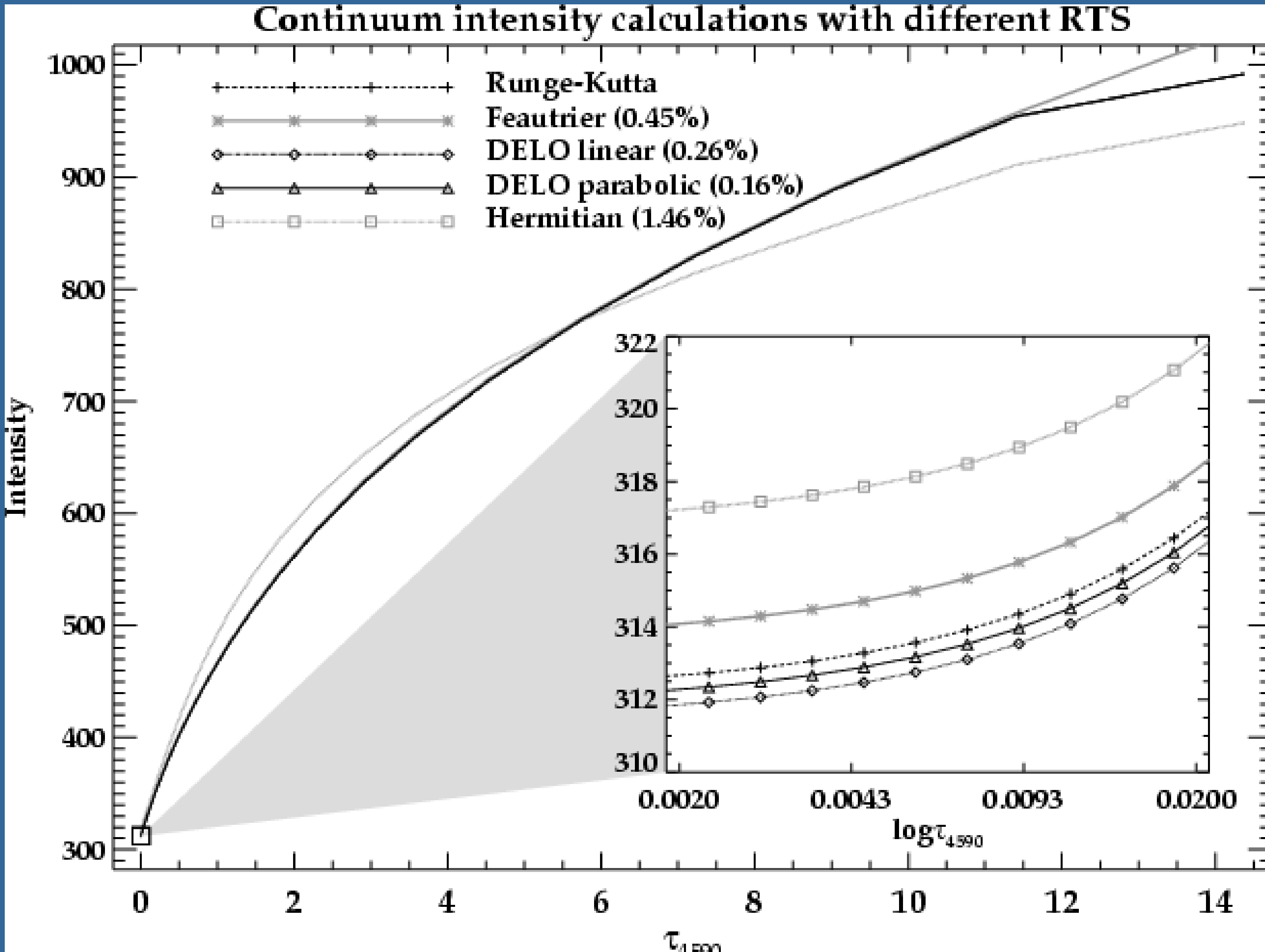
$$I_{i+1} = I_i + \frac{\delta_i}{2}\left(I_i' + I_{i+1}'\right) + \frac{\delta_i^2}{12}\left(I_i'' - I_{i+1}''\right)$$

$$I' = k\rho \cdot (I - S)$$

$$I'' = k\rho \cdot \left[k\rho \cdot (I - S) - S'\right] + (k\rho)' \cdot (I - S)$$

$$I_{i+1} = \alpha_i \cdot I_i + \beta_i$$

Comparison of the solvers



Home work

Compute spectral synthesis using a method of your choice for a static 1D model atmosphere of the Sun. For a fixed geometrical depth grid and wavelength grid you are give a 2D array of opacities and 2D array of source function. The boundary conditions: no radiation enters through the surface and the flux spectrum at the deepest point is given.