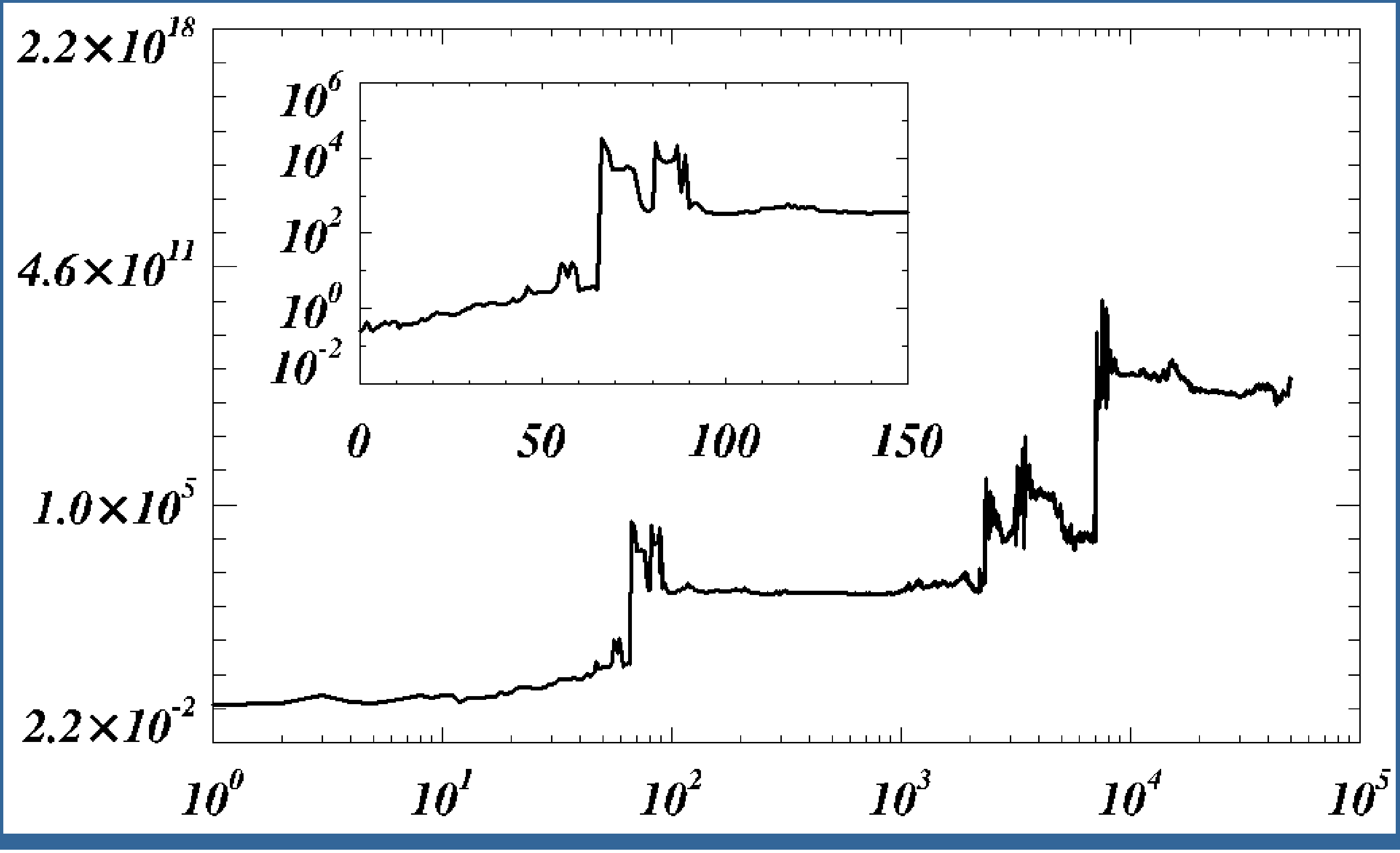
Home Work #1 (problem 1)

Inversion results:



HW #1 (problem 3)

Analysis of the equation:

$$\frac{dy}{dx} = y + e^x; \qquad y\Big|_{x=0} = 1; \qquad y\Big|_{x=50} = 1000;$$

$$y = \left(\tfrac{1}{2}\cdot x - \tfrac{1}{4} + a\right)\cdot e^x + b\cdot e^{-x}$$

$$\begin{cases} a + b = \frac{5}{4} \\ a + b\cdot e^{-100} = 1000\cdot e^{-50} - \frac{99}{4} \end{cases} \xrightarrow{\approx} \begin{cases} a = 1000\cdot e^{-50} - \frac{99}{4} \\ b = \frac{104}{4} \end{cases}$$

RK4 with adaptive step

- System of ODE:

$$\frac{dy_1}{dx} = y_2; \qquad \frac{dy_2}{dx} = y_1 + e^x$$

- Aiming parameter (AP):

$$\{y_1, y_2\}\Big|_{x=0} = \{1, \text{ aim high } (\ y_1\big|_{x=50} > 1000)\}$$

$$\{y_1, y_2\}\Big|_{x=0} = \{1, \text{ aim low } (\ y_1\big|_{x=50} < 1000)\}$$

Problem: AP is ≈2, change in AP of 10^{-12} changes $y_1(50)$ by 10^{20}

Solutions

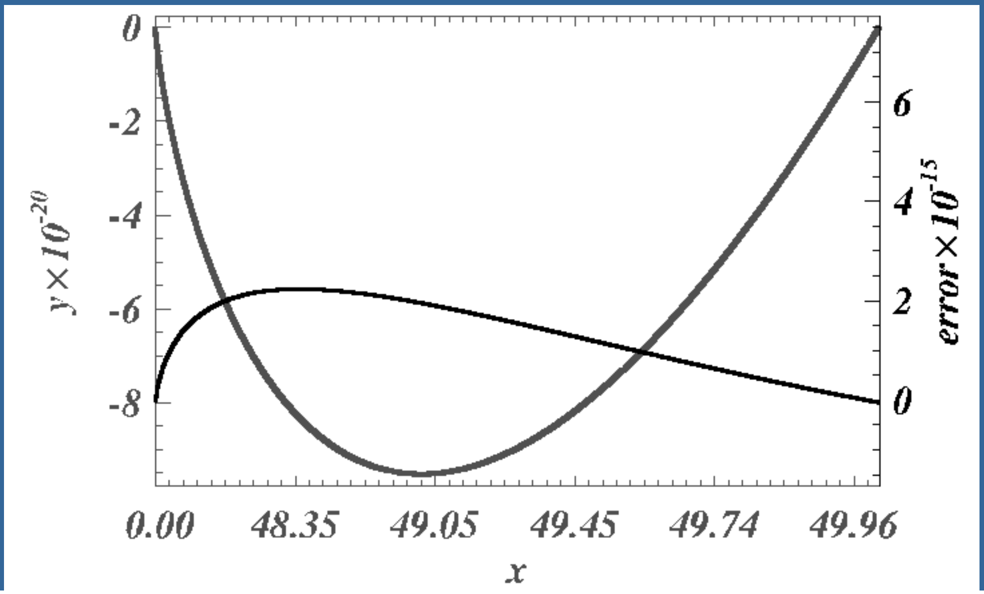
- ***Algorithm A***:
  - Aim from x=0
  - Aim from x=50
  - Take $y = (50 - x) / 50 \cdot y^{\text{left}} + x/50 \cdot y^{\text{right}}$
- ***Algorithm B***: replace variable $t = e^x$

Finite differences

Straight forward:

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ 0 & 1 & -2-\Delta^2 & 1 & 0 \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \times \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} y^{\text{left}} \\ \vdots \\ \Delta^2 e^{k\Delta} \\ \vdots \\ y^{\text{right}} \end{pmatrix}$$

Solution:



Self-consistent solution of RT

- Take into account the intensity part of the source function (scattering)
- Construct self-consistent solution for the intensity and the source function (Λ-iterations)
- Efficiency of the process in different situations
- Speeding things up

Source function again

Source function is a ratio of emission to absorption:

$$S_\nu(x,\mu) = \frac{j_\nu(x,\mu)}{k_\nu(x,\mu)}$$

The emission consists of two parts: *stimulated* ($\propto$ to local intensity) and *spontaneous* (function of the local conditions):

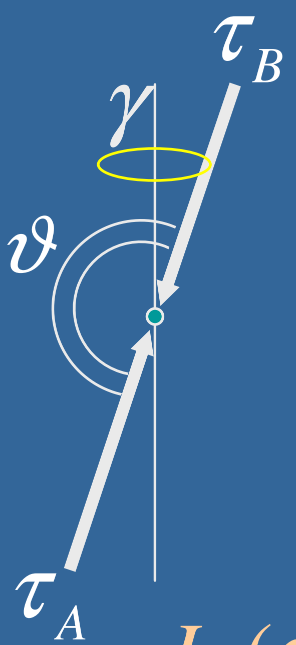
$$S_\nu(x,\mu) = \frac{j_\nu^{\text{stim}}(x,\mu) + j_\nu^{\text{spont}}(x,\mu)}{k_\nu(x,\mu)} =$$

$$= (1-\varepsilon)\cdot J_\nu(x) + \varepsilon \cdot B_\nu$$

RT for a *b-b* transition

A complete set of equations for a single *b-b* transition:

$$\frac{dI_\nu(\tau,\gamma,\vartheta)}{d\tau} = I_\nu(\tau,\gamma,\vartheta) - S_\nu(\tau,\gamma,\vartheta)$$



$$S_\nu(\tau,\gamma,\vartheta) = \tfrac{1-\varepsilon}{4\pi} \int_{-\infty}^{\infty} \varphi(\nu-\nu')d\nu' \oiint I_{\nu'} d\gamma d\vartheta + \varepsilon \mathrm{B}_\nu(T)$$

Now we can write the formal solution:

$$I_\nu(\tau,\gamma,\vartheta) = \begin{cases} \displaystyle\int_{\tau}^{\tau_B} s_\nu(t,\gamma,\vartheta)\cdot e^{-(t-\tau)}dt + e^{-(\tau_B-\tau)}\cdot I_\nu^B \\ \displaystyle\int_{\tau_A}^{\tau} s_\nu(t,\gamma,\vartheta)\cdot e^{-(\tau-t)}dt + e^{-(\tau-\tau_A)}\cdot I_\nu^A \end{cases}$$

Formal solutions: coordinate convention

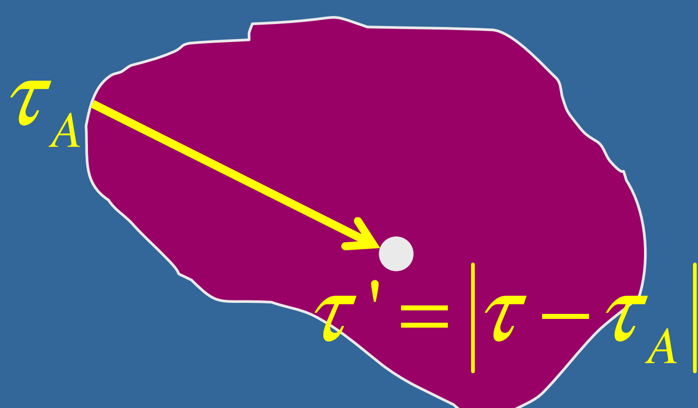
- Source function:

$$S_\nu(\tau,\gamma,\vartheta)=\tfrac{1-\varepsilon}{4\pi}\int_{-\infty}^{\infty}\varphi(\nu-\nu')d\nu'\oiint I_{\nu'}d\gamma d\vartheta+$$

$$+\varepsilon \mathrm{B}_\nu(T)$$

- Intensity: $I_\nu(\tau',\gamma,\vartheta)=e^{-\tau'}\cdot I_\nu^A+$

$$+\int_0^{\tau'} S_\nu(t,\gamma,\vartheta)\cdot e^{-t}\cdot dt$$



Now we can combine formal solutions in one self-consistent integral equation:

$$S_\nu(\tau) = (1-\varepsilon) \oint\!\!\oint \frac{d\Omega}{4\pi} \int_0^\tau dt \int_{-\infty}^{\infty} d\nu' \cdot \varphi(\nu-\nu') \cdot e^{-(\tau-t)} \cdot S_{\nu'}(t) +$$

$$+ (1-\varepsilon) e^{-\tau} \boxed{\oint\!\!\oint \frac{d\Omega}{4\pi} \int_{-\infty}^{\infty} d\nu' \cdot \varphi(\nu-\nu') \cdot I_{\nu'}^{\text{bound}}} + \varepsilon \, \mathrm{B}_\nu(T)$$

We can re-write this in operator form also known as Λ (lambda) operator

$$S_\nu^{\text{bound}}$$

$$\Lambda = \oint\!\!\oint \frac{d\Omega}{4\pi} \int_0^\tau \int_{-\infty}^{\infty} \varphi(\nu-\nu') \cdot e^{-(\tau-t)} \cdot d\nu' \cdot dt$$

$$S_\nu(\tau) = (1-\varepsilon)\Lambda S_\nu + \varepsilon \, \mathrm{B}_\nu(T) + (1-\varepsilon) \cdot e^{-\tau} \cdot S_\nu^{\text{bound}}$$

Λ -operator is linear:

$$\Lambda(\alpha \cdot S_1 + \beta \cdot S_2) = \alpha \cdot \Lambda S_1 + \beta \cdot \Lambda S_2$$

Λ-iterations

- Recurrent relation:

$$S_0 = \varepsilon \mathrm{B}_\nu$$

$$S_{i+1} = (1-\varepsilon)\Lambda S_i + \varepsilon \mathrm{B}_\nu + (1-\varepsilon)e^{-\tau} S_\nu^{\text{bound}}$$

- Convergence rate:

$$S^{n+1} = (1-\varepsilon)\Lambda S^n + \varepsilon \mathrm{B} + (1-\varepsilon)e^{-\tau} S_\nu^{\text{bound}}$$

$$-\quad S^n = (1-\varepsilon)\Lambda S^{n-1} + \varepsilon \mathrm{B} + (1-\varepsilon)e^{-\tau} S_\nu^{\text{bound}}$$

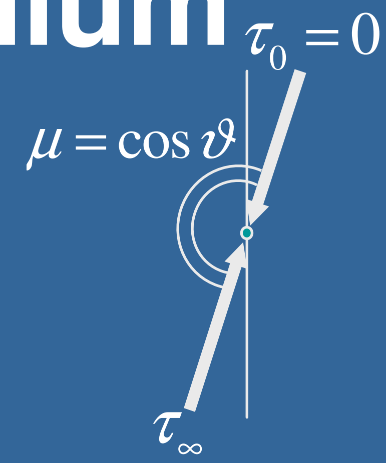
$$\Delta S^n = (1-\varepsilon)\Lambda(\Delta S^{n-1}) \qquad (\tau > 1)$$

Convergence etc.

- Convergence is slow for small ε and τ >1.

  $$\Delta S^{n} = (1-\varepsilon)\Lambda(\Delta S^{n-1})$$
- Convergence is monotonous (corrections have always the same sign).
- Note that: $\Lambda S_\nu = J_\nu$
- We could also substitute the expression for the source function to the expression for intensity. This is useful when we need to include the radiative energy transport to hydrodynamics.
- Home work: *write the expression for Λ-operator for angle averaged intensity* J_ν

1D semi-infinite medium



Given one selected direction:

$$\mu \frac{dI_\nu(\tau,\mu)}{d\tau} = I_\nu(\tau,\mu) - S_\nu(\tau)$$

$$S_\nu(\tau,\mu) = \tfrac{1-\varepsilon}{2} \int_{-\infty}^{\infty} \varphi(\nu-\nu')d\nu' \int_{-1}^{1} I_{\nu'}(\tau,\mu)\,d\mu + \varepsilon\, \mathrm{B}_\nu(T)$$

Now we can write the formal solution:

$$I_\nu(\tau,\mu) = \begin{cases} \frac{1}{\mu}\int_\tau^\infty s_\nu(t)\cdot e^{-\frac{(t-\tau)}{\mu}}\,dt, & \mu > 0 \\ -\frac{1}{\mu}\int_0^\tau s_\nu(t)\cdot e^{-\frac{(t-\tau)}{\mu}}\,dt, & \mu < 0 \end{cases}$$

Now we can write a self consistent equation for the source function in 1D:

$$S_\nu(\tau) = \tfrac{1-\varepsilon}{2}\left[\int_0^1 \frac{d\mu}{\mu}\int_\tau^\infty \int_{-\infty}^\infty \varphi(\nu-\nu')d\nu' \cdot e^{-\frac{(t-\tau)}{\mu}}dt \cdot S_{\nu'}(t) - \right.$$

$$\left.\int_{-1}^0 \frac{d\mu}{\mu}\int_0^\tau \int_{-\infty}^\infty \varphi(\nu-\nu')d\nu' \cdot e^{-\frac{(t-\tau)}{\mu}}dt \cdot S_{\nu'}(t)\right] + \varepsilon \mathrm{B}_\nu(T)$$

$$\Lambda = \tfrac{1}{2}\left[\int_0^1 \frac{d\mu}{\mu}\int_\tau^\infty \int_{-\infty}^\infty \varphi(\nu-\nu')d\nu' \cdot e^{-\frac{(t-\tau)}{\mu}}dt - \right.$$

$$\left.\int_{-1}^0 \frac{d\mu}{\mu}\int_0^\tau \int_{-\infty}^\infty \varphi(\nu-\nu')d\nu' \cdot e^{-\frac{(t-\tau)}{\mu}}dt\right]$$

$$S_\nu(\tau) = (1-\varepsilon)\Lambda S_\nu + \varepsilon \mathrm{B}_\nu(T)$$

Accelerated Λ-iterations

- Λ-operator is a convolution with non-negative kernel in frequency and spatial domains. Numerically it can be approximated with summation:

$$\Lambda S_\nu \approx \sum_{\nu,\tau} \omega_{\nu,\tau} s_\nu(\tau) = \sum_k \omega_{j,k} s_k$$

- Newton-Raphson acceleration (Cannon)

$$S - (1-\varepsilon)\Lambda S - \varepsilon B = \Phi(S) = 0$$

- Operator splitting (Scharmer, Olson et al.)

$$S^{(l+1)} = (1-\varepsilon)\Lambda S^{(l)} + \varepsilon B$$

$$\Lambda = \Lambda^* + (\Lambda - \Lambda^*)$$

The recurrent relation for lambda iterations (Cannon):

$$S_\nu^{(l+1)}(\tau) - (1-\varepsilon)\Lambda S^{(l)} - \varepsilon\,\mathrm{B}_\nu(\tau) = 0$$

$$S_j^{(l+1)} = (1-\varepsilon_j)\sum_{k=1}^{N}\omega_{jk}S_k^{(l)} + \varepsilon_j B_j$$

$$\Delta S_j^{(l)} = (1-\varepsilon_j)\sum_{k=1}^{N}\omega_{jk}S_k^{(l)} + \varepsilon_j B_j - S_j^{(l)} = \Phi(S_k^{(l)})$$

$$S_j^{(l+1)} = S_j^{(l)} - \frac{\Phi(S_k^{(l)})}{d\Phi(S_k^{(l)})\Big/ dS_j^{(l)}} =$$

$$= \frac{(1-\varepsilon_j)\sum_{k\neq j}^{N}\omega_{jk}S_k^{(l)} + \varepsilon_j B_{\nu j}}{(1-\varepsilon_j)\omega_{jj} - 1}$$

Operator splitting (Scharmer, Olson):

$$S^{(l)} + \delta S^{(l)} = (1-\varepsilon)\Lambda(S^{(l)} + \delta S^{(l)}) + \varepsilon \mathrm{B}$$

$$\delta S^{(l)} \equiv S^{(l+1)} - S^{(l)}$$

$$\Lambda = \Lambda^* + (\Lambda - \Lambda^*)$$

$$S^{(l)} + \delta S^{(l)} = (1-\varepsilon)\Lambda^* S^{(l)} + \underline{(1-\varepsilon)\Lambda^*\delta S^{(l)}} +$$

$$+ \underline{(1-\varepsilon)\Lambda S^{(l)}} + \underline{\underline{(1-\varepsilon)\Lambda\delta S^{(l)}}} -$$

$$- (1-\varepsilon)\Lambda^* S^{(l)} - \underline{\underline{(1-\varepsilon)\Lambda^*\delta S^{(l)}}} + \varepsilon \mathrm{B}$$

$$S^{(l)} + \delta S^{(l)} = (1-\varepsilon)\Lambda^*\delta S^{(l)} + (1-\varepsilon)\Lambda S^{(l)} +$$

$$\underline{\underline{+ (1-\varepsilon)[\Lambda - \Lambda^*]\delta S^{(l)}}} + \varepsilon \mathrm{B}$$

$$\delta S^{(l)} - (1-\varepsilon)\Lambda^*\delta S^{(l)} \approx (1-\varepsilon)\Lambda S^{(l)} + \varepsilon \mathrm{B} - S^{(l)}$$

$$\delta S^{(l)} = [1 - (1-\varepsilon)\Lambda^*]^{-1} \cdot [(1-\varepsilon)\Lambda S^{(l)} + \varepsilon \mathrm{B} - S^{(l)}]$$

More general assessment of what was done:

- Λ-iterations are used to find source function in case of radiation-dependent absorption/emission
- The direct iteration scheme for the source function can be used but the convergence is slow, especially at high optical depths
- Lambda integral in ν and τ can be replaced with a quadrature formula using a discrete grid
- We have replaced the true lambda operator with an approximate operator that has better convergence properties (higher order convergence) and can be easily inverted
- The freedom in choice of the accelerated lambda operator helps to optimize it for special cases