PART I: INTRODUCTION

Radiation and Matter

Section 1

Radiation

Almost all the astrophysical information we can derive about distant sources results from the radiation that reaches us from them. Our starting point is, therefore, a review of the principal ways of describing radiation. (In principle, this could include polarization properties, but we neglect that for simplicity).

The fundamental definitions of interest are of (specific) *intensity* and (physical) *flux.*

1.1 Specific Intensity, I_ν

The specific intensity (or radiation intensity, or surface brightness) is defined as:

> the rate of energy flowing at a given point,
> per unit area,
> per unit time,
> per unit frequency interval,
> per unit solid angle (in azimuth ϕ and direction θ to the normal; refer to the geometry sketched in Fig. 1.1)

or, expressed algebraically,

$$\begin{aligned} I_\nu(\theta,\phi) &= \frac{\mathrm{d}E_\nu}{\mathrm{d}S\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega} \\ &= \frac{\mathrm{d}E_\nu}{\mathrm{d}A\cos\theta\;\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega} \quad [\mathrm{J\ m^{-2}\ s^{-1}\ Hz^{-1}\ sr^{-1}}]. \end{aligned} \tag{1.1}$$

We've given a 'per unit frequency definition', but we can always switch to 'per unit wavelength' by noting that, for some frequency-dependent physical quantity 'X', we can write

$$X_\nu\,\mathrm{d}\nu = X_\lambda\,\mathrm{d}\lambda.$$

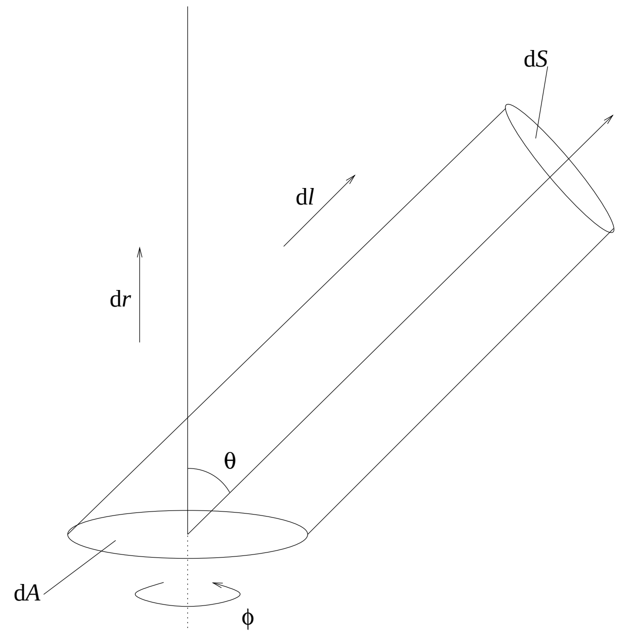


Figure 1.1: Geometry used to define radiation quantities. The element of area dA might, for example, be on the surface of a star.

(which has the same dimensionality on each side of the equation). Mathematically, $\mathrm{d}\nu/\,\mathrm{d}\lambda = -c/\lambda^2$, but physically this just reflects the fact that increasing frequency means decreasing wavelength. We clearly require a positive physical quantity on either side of the equation, so specific intensity per unit *wavelength* is related to I_ν by

$$I_\lambda = I_\nu \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = I_\nu \frac{c}{\lambda^2} \quad [\mathrm{J\ m^{-2}\ s^{-1}\ m^{-1}\ sr^{-1}}]$$

where the θ, ϕ dependences are implicit (as will generally be the case; the sharp-eyed will note also that we appear to have 'lost' a minus sign in evaluating $\mathrm{d}\nu/\mathrm{d}\lambda$, but this just is because frequency increases as wavelength decreases). Equation (1.1) defines the *monochromatic* specific intensity ('monochromatic' will usually also be implicit); we can define a *total* intensity by integrating over frequency:

$$I = \int_0^\infty I_\nu \,\mathrm{d}\nu \quad [\mathrm{J\ m^{-2}\ s^{-1}\ sr^{-1}}].$$

1.1.1 Mean Intensity, J_ν

The mean intensity is, as the name suggests, the average of I_ν over solid angle; it is of use when evaluating the rates of physical processes that are photon dominated but independent of the angular distribution of the radiation (e.g., photoionization and photoexcitation rates).

$$J_\nu = \frac{\int_\Omega I_\nu \, \mathrm{d}\Omega}{\int \mathrm{d}\Omega} = \frac{1}{4\pi}\int_\Omega I_\nu \, \mathrm{d}\Omega$$
$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_\nu \, \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \quad [\mathrm{J\ m^{-2}\ s^{-1}\ Hz^{-1}\ sr^{-1}}] \tag{1.2}$$

since

$$\int_\Omega \mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{1.3}$$

Introducing the standard astronomical nomenclature $\mu = \cos\theta$ (whence $\mathrm{d}\mu = -\sin\theta \, \mathrm{d}\theta$), we have

$$\int \mathrm{d}\Omega = \left(-\int_0^{2\pi}\int_{+1}^{-1} \mathrm{d}\mu \, \mathrm{d}\phi = \right) \int_0^{2\pi}\int_{-1}^{+1} \mathrm{d}\mu \, \mathrm{d}\phi \tag{1.4}$$

and eqtn. (1.2) becomes

$$J_\nu = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} I_\nu(\mu, \phi) \, \mathrm{d}\mu \, \mathrm{d}\phi \tag{1.5}$$

(where for clarity we show the μ, ϕ dependences of I_ν explicitly).

If the radiation field is independent of ϕ but not θ (as in the case of a stellar atmosphere without starspots, for example) then this simplifies to

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu) \, \mathrm{d}\mu. \tag{1.6}$$

From this it is evident that if I_ν is *completely* isotropic (i.e., no $\theta[\equiv \mu]$ dependence, as well as no ϕ dependence), then $J_\nu = I_\nu$. (This should be intuitively obvious – if the intensity is the same in all directions, then the mean intensity must equal the intensity [in any direction].)

1.2 Physical Flux, F_ν

The physical flux (or radiation flux density, or radiation flux, or just 'flux') is the net rate of energy flowing across unit area (e.g., at a detector), *from all directions*, per unit time, per unit frequency interval:

$$F_\nu = \frac{\int_\Omega \mathrm{d}E_\nu}{\mathrm{d}A \, \mathrm{d}t \, \mathrm{d}\nu} \quad [\mathrm{J\ m^{-2}\ s^{-1}\ Hz^{-1}}]$$

It is the absence of directionality that crucially distinguishes *flux* from *intensity,* but the two are clearly related. Using eqtn. (1.1) we see that

$$F_\nu = \int_\Omega I_\nu \cos\theta \,\mathrm{d}\Omega \tag{1.7}$$

$$= \int_0^{2\pi}\int_0^{\pi} I_\nu \cos\theta \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \quad [\mathrm{J\ m^{-2}\ Hz^{-1}}] \tag{1.8}$$

$$= \int_0^{2\pi}\int_{-1}^{+1} I_\nu(\mu,\phi)\mu \,\mathrm{d}\mu \,\mathrm{d}\phi$$

or, if there is no ϕ dependence,

$$F_\nu = 2\pi \int_{-1}^{+1} I_\nu(\mu)\mu \,\mathrm{d}\mu. \tag{1.9}$$

Because we're simply measuring the energy flowing across an area, there's no explicit directionality involved – other than if the energy impinges on the area from 'above' or 'below'.[1] It's therefore often convenient to divide the contributions to the flux into the 'upward' (emitted, or 'outward') radiation ($F^+_\nu; 0 \leq \theta \leq \pi/2$, Fig 1.1) and the 'downward' (incident, or 'inward') radiation ($F^-_\nu; \pi/2 \leq \theta \leq \pi$), with the net upward flux being $F_\nu = F^+_\nu - F^-_\nu$:

$$F_\nu = \int_0^{2\pi}\int_0^{\pi/2} I_\nu \cos\theta \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \qquad + \int_0^{2\pi}\int_{\pi/2}^{\pi} I_\nu \cos\theta \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi$$

$$\equiv F^+_\nu \qquad\qquad - F^-_\nu$$

As an important example, the surface flux emitted by a star is just F^+_ν (assuming there is no incident external radiation field);

$$F_\nu = F^+_\nu = \int_0^{2\pi}\int_0^{\pi/2} I_\nu \cos\theta \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi$$

or, if there is no ϕ dependence,

$$= 2\pi \int_0^{\pi/2} I_\nu \cos\theta \sin\theta \,\mathrm{d}\theta.$$

$$= 2\pi \int_0^{+1} I_\nu(\mu)\mu \,\mathrm{d}\mu. \tag{1.10}$$

[1] In principle, flux is a vector quantity, but the directionality is almost always implicit in astrophysical situations; e.g., from the centre of a star outwards, or from a source to an observer.

If, furthermore, I_ν has no θ dependence *over the range 0–π/2* then

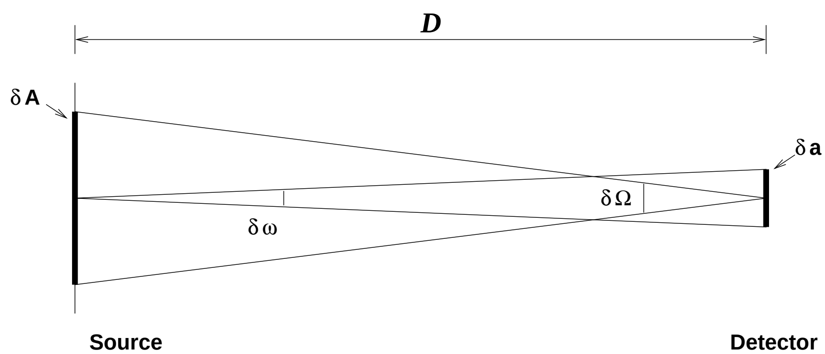
$$F_\nu = \pi I_\nu \tag{1.11}$$

(since $\int_0^{\pi/2} \cos\theta \, \sin\theta \, \mathrm{d}\theta = 1/2$). If I_ν is *completely* isotropic, then $F_\nu^+ = F_\nu^-$, and $F_\nu = 0$.

1.3 Flux vs. Intensity

A crucial difference between I_ν and F_ν should be noted: the *specific intensity is independent of distance from the source* (but requires the source to be resolved), while the *physical flux falls off as r^{-2}*.

This can be understood by noting that specific intensity is defined in terms of 'the rate of energy flow per unit area of surface... per unit solid angle'. The energy flow per unit area falls off as r^{-2}, but the area per unit solid angle increases as r^2, and so the two cancel.



Expressing this formally: suppose some area δA on a source at distance D subtends a solid angle $\delta\Omega$ at a detector; while the detector, area δa, subtends a solid angle $\delta\omega$ at the source. The energy emitted towards (and received by) the detector is

$$E = I_\nu \, \delta A \, \delta\omega; \text{ but}$$
$$\delta A = D^2 \delta\Omega \text{ and } \delta\omega = \delta a / D^2, \text{ so}$$
$$\frac{E}{\delta\Omega} = I_\nu D^2 \frac{\delta a}{D^2};$$

that is, the energy received *per unit solid angle* (i.e., the intensity) is distance independent. Equivalently, we can say that the *surface brightness* of source is distance independent (in the absence of additional processes, such as interstellar extinction).

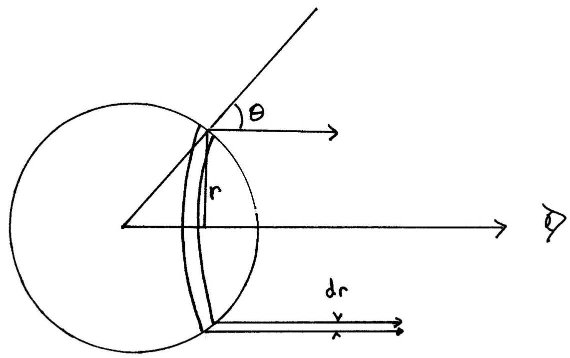
A source must be spatially resolved for us to be able to measure the intensity; otherwise, we can measure 'only' the flux – if the source is unresolved, we can't identify different directions towards it. Any spatially extended source will, at some large enough distance D, produce an image source at the focal plane of a telescope that will be smaller than the detector (pixel) size. For such an unresolved source, the detected energy is

$$E = I_\nu \, \delta a \, \delta\Omega$$
$$= I_\nu \, \delta a \, \frac{\delta A}{D^2}$$

and we recover the expected inverse-square law for the detected flux.

1.3.1 Flux from a star

To elaborate this, consider the flux from a star at distance D.



The observer sees the projected area of the annulus as

$$\mathrm{d}A = 2\pi r\,\mathrm{d}r$$

and since

$$r = R\,\sin\theta \qquad (\mathrm{d}r = R\cos\theta\,\mathrm{d}\theta)$$

we have

$$\begin{aligned}\mathrm{d}A &= 2\pi R\sin\theta\;R\cos\theta\,\mathrm{d}\theta\\ &= 2\pi R^2\sin\theta\cos\theta\,\mathrm{d}\theta\\ &= 2\pi R^2\mu\,\mathrm{d}\mu\end{aligned}$$

where as usual $\mu = \cos\theta$. The annulus therefore subtends a solid angle

$$\mathrm{d}\Omega = \frac{\mathrm{d}A}{D^2} = 2\pi\left(\frac{R}{D}\right)^2\mu\,\mathrm{d}\mu.$$

The flux received from this solid angle is

$$\mathrm{d}f_\nu = I_\nu(\mu)\,\mathrm{d}\Omega$$

so that the total observed flux is

$$f_\nu = 2\pi\left(\frac{R}{D}\right)^2\int_0^1 I_\nu\mu\,\mathrm{d}\mu$$

or, using eqtn. (1.10),

$$\begin{aligned}&= \left(\frac{R}{D}\right)^2 F_\nu\\ &= \theta_*^2 F_\nu \qquad [\mathrm{J\ m^{-2}\ s^{-1}\ Hz^{-1}}]\end{aligned}$$

where θ_* is the solid angle subtended by the star (measured in radians).

1.4 Flux Moments

Flux moments are a traditional 'radiation' topic, of use in studying the transport of radiation in stellar atmospheres. The $n^{\rm th}$ moment of the radiation field is defined as

$$M_\nu \equiv \frac{1}{2}\int\limits_{-1}^{+1} I_\nu(\mu)\mu^n \,\mathrm{d}\mu. \tag{1.12}$$

We can see that we've already encountered the *zeroth-order moment*, which is the mean intensity:

$$J_\nu = \frac{1}{2}\int\limits_{-1}^{+1} I_\nu(\mu) \,\mathrm{d}\mu. \tag{1.6}$$

We have previously written the flux as

$$F_\nu = 2\pi \int\limits_{-1}^{+1} I_\nu(\mu)\mu \,\mathrm{d}\mu; \tag{1.9}$$

to cast this in the same form as eqtns. (1.12) and (1.6), we define the 'Eddington flux' as $H_\nu = F_\nu/(4\pi)$, i.e.,

$$H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\mu \,\mathrm{d}\mu. \tag{1.13}$$

We see that H_ν is the *first-order moment* of the radiation field.

The second-order moment, the so-called 'K integral', is, from the definition of moments,

$$K_\nu = \frac{1}{2}\int\limits_{-1}^{+1} I_\nu(\mu)\mu^2 \,\mathrm{d}\mu \tag{1.14}$$

In the special case that I_ν is isotropic we can take it out of the integration over μ, and

$$\begin{aligned} K_\nu &= \frac{1}{2}\frac{\mu^3}{3}I_\nu\Big|_{-1}^{+1} \\ &= \frac{1}{3}I_\nu \qquad \left[\text{also} = \frac{1}{3}J_\nu \text{ for isotropy}\right] \end{aligned} \tag{1.15}$$

We will see in Section 1.8 that the K integral is straightforwardly related to radiation pressure.

Higher-order moments are rarely used. So, to recap (and using the notation first introduced by Eddington himself), for $n = 0, 1, 2$:

$n = 0$	Mean Intensity	$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\,\mathrm{d}\mu$
$n = 1$	Eddington flux	$H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\mu\,\mathrm{d}\mu$
$n = 2$	K integral	$K_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\mu^2\,\mathrm{d}\mu$

(all with units [J m^{-2} s^{-1} Hz^{-1} sr^{-1}]).

We can also define the integral quantities

$$\mathrm{J} = \int_0^\infty J_\nu \, \mathrm{d}\nu$$

$$\mathrm{F} = \int_0^\infty F_\nu \, \mathrm{d}\nu$$

$$\mathrm{K} = \int_0^\infty K_\nu \, \mathrm{d}\nu$$

1.5 Other 'Fluxes', 'Intensities'

Astronomers can be rather careless in their use of the terms 'flux'. and 'intensity'. The 'fluxes' and 'intensities' discussed so far can all be quantified in terms of physical (e.g., SI) units. Often, however, astronomical signals are measured in more arbitrary ways (such 'integrated signal at the detector', or even 'photographic density'); in such cases, it's commonplace to refer to the 'intensity' in a spectrum, but this is just a loose shorthand, and doesn't allude to the true specific intensity defined in this section.

There are other physically-based quantities that one should be aware of. For example, discussions of model stellar atmospheres may refer to the *'astrophysical flux'*; this is given by F_ν/π (also called, rarely, the 'radiative flux'), which is evidently similar to the Eddington flux, $H_\nu = F_\nu/(4\pi)$, which has itself also occasionally been referred to as the 'Harvard flux'. Confusingly, some authors also call it just 'the flux', but it's always written as H_ν (never F_ν).

1.6 Black-body radiation (reference/revision only)

In astrophysics, a radiation field can often be usefully approximated by that of a 'black body', for which the intensity is given by the Planck function:

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}\left\{\exp\left(\frac{h\nu}{kT}\right) - 1\right\}^{-1} \quad [\mathrm{J\ m^{-2}\ s^{-1}\ Hz^{-1}\ sr^{-1}}];\ \text{or} \tag{1.16}$$

$$I_\lambda = B_\lambda(T) = \frac{2hc^2}{\lambda^5}\left\{\exp\left(\frac{hc}{\lambda kT}\right) - 1\right\}^{-1} \quad [\mathrm{J\ m^{-2}\ s^{-1}\ m^{-1}\ sr^{-1}}] \tag{1.17}$$

(where $B_\nu \, \mathrm{d}\nu = B_\lambda \, \mathrm{d}\lambda$).

We have seen that

$$F_\nu = 2\pi \int_{-1}^{+1} I_\nu(\mu)\mu \, \mathrm{d}\mu. \tag{1.9}$$

If we have a surface radiating like a black body then $I_\nu = B_\nu(T)$, and there is no μ dependence, other than that the energy is emitted over the limits $0 \leq \mu \leq 1$; thus the physical flux for a black-body radiator is given by

$$F_\nu = F_\nu^+ \quad = 2\pi \int\limits_0^{+1} B_\nu(T)\mu \,\mathrm{d}\mu \quad = B_\nu \left.\frac{2\pi\mu^2}{2}\right|_0^{+1}$$
$$= \pi B_\nu. \tag{1.18}$$

(cp. eqtn. (1.11): $F_\nu = \pi I_\nu$)

1.6.1 Integrated flux

The total radiant energy flux is obtained by integrating eqtn. (1.18) over frequency,

$$\int_0^\infty F_\nu \,\mathrm{d}\nu = \int_0^\infty \pi B_\nu \,\mathrm{d}\nu$$
$$= \int_0^\infty \frac{2\pi h\nu^3}{c^2} \left\{\exp\left(\frac{h\nu}{kT}\right) - 1\right\}^{-1} \mathrm{d}\nu. \tag{1.19}$$

We can solve this by setting $x = (h\nu)/(kT)$ (whence $\mathrm{d}\nu = [{}^{kT}/{}_h]\ \mathrm{d}x$), so

$$\int_0^\infty F_\nu \,\mathrm{d}\nu = \left(\frac{kT}{h}\right)^4 \frac{2\pi h}{c^2} \int_0^\infty \frac{x^3}{\exp(x) - 1}\,\mathrm{d}x$$

The integral is now a standard form, which has the solution $\pi^4/15$, whence

$$\int_0^\infty F_\nu \,\mathrm{d}\nu = \left(\frac{k\pi}{h}\right)^4 \frac{2\pi h}{15c^2} T^4 \tag{1.20}$$
$$\equiv \sigma T^4 \tag{1.21}$$

where σ is the Stefan-Boltzmann constant,

$$\sigma = \frac{2\pi^5 k^4}{15h^3c^2} = 5.67 \times 10^{-5} \quad [\mathrm{J\ m^{-2}\ K^{-4}\ s^{-1}}].$$

1.6.2 Approximate forms

There are two important approximations to the Planck function which follow directly from eqtn. 1.16:

$$B_\nu(T) \simeq \frac{2h\nu^3}{c^2}\left\{\exp\left(\frac{h\nu}{kT}\right)\right\}^{-1} \qquad \text{for } \frac{h\nu}{kT} \gg 1 \tag{1.22}$$

(Wien approximation), and

$$B_\nu(T) \simeq \frac{2\nu^2 kT}{c^2} \qquad \text{for } \frac{h\nu}{kT} \ll 1 \tag{1.23}$$

(Rayleigh-Jeans approximation; $\exp(h\nu/kT) \simeq 1 + h\nu/kT$).

The corresponding wavelength-dependent versions are, respectively,

$$B_\lambda(T) \simeq \frac{2hc^2}{\lambda^5}\left\{\exp\left(\frac{hc}{\lambda kT}\right)\right\}^{-1},$$
$$B_\lambda(T) \simeq \frac{2ckT}{\lambda^4}.$$

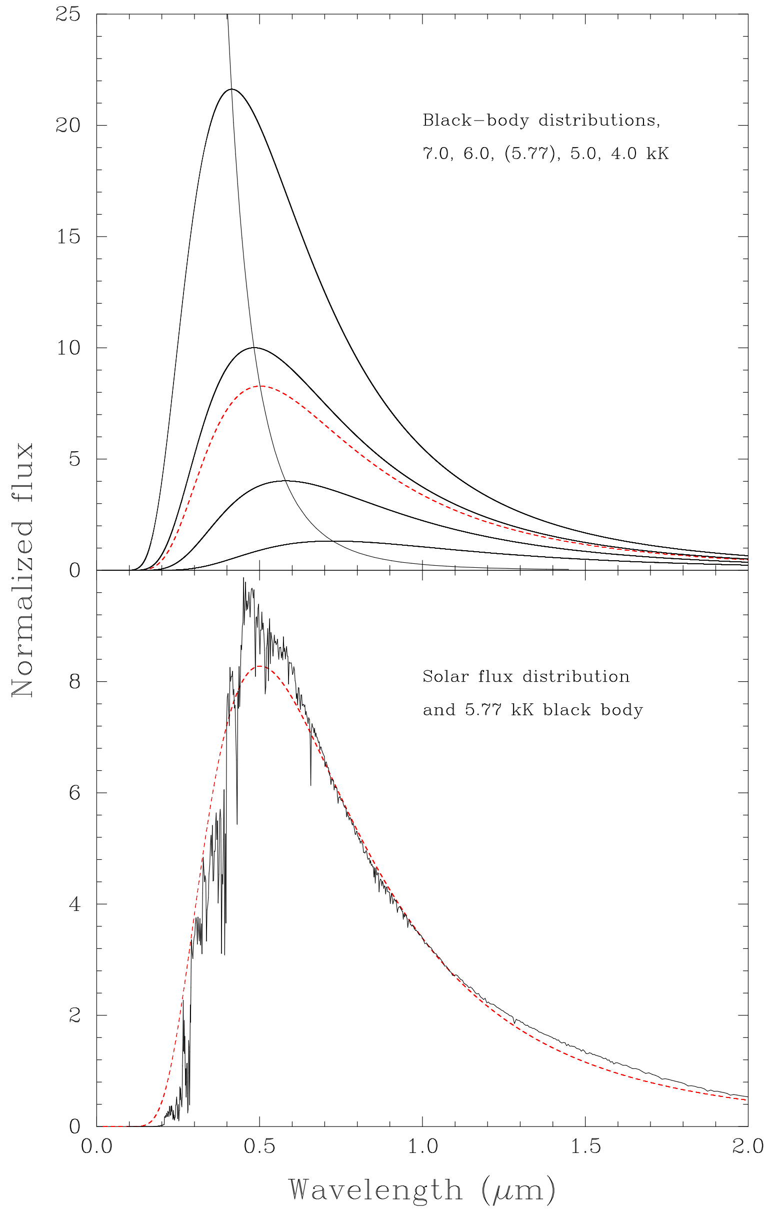


Figure 1.2: *Upper panel:* Flux distributions for black bodies at several different temperatures. A hotter black body radiates more energy at *all* wavelengths than a cooler one, but the increase is greater at shorter wavelengths. The peak of the black-body distribution migrates blueward with increasing temperature, in accordance with Wien's law (also plotted).

Lower panel: Flux distribution for the Sun (actually, a Kurucz solar model) compared with a black-body distribution at the same temperature. The black body is a reasonable, though far from perfect, match to the model, the main differences arising because of line blocking in the sun at short wavelengths. This energy must come out elsewhere, and appears as an excess over the black body at long wavelengths.

(Flux units are 10^7 J m^{-2} s^{-1} μm^{-1}.)

The Wien approximation to the Planck function is very good at wavelengths shortwards of and up to the peak of the flux distribution; but one generally needs to go something like $\sim 10\times$ the peak wavelength before the long-wavelength Rayleigh-Jeans approximation is satisfactory.

1.6.3 Wien's Law

Wien's displacement law (not to be confused with the Wien approximation!) relates the black-body temperature to the wavelength of peak emission. To find the peak, we differentiate eqtn. (1.17) with respect to wavelength, and set to zero:

$$\frac{\partial B}{\partial \lambda} = 8hc\left(\frac{hc}{\lambda^7 kT}\frac{\exp\{hc/\lambda kT\}}{(\exp\{hc/\lambda kT\}-1)^2} - \frac{1}{\lambda^6}\frac{5}{\exp\{hc/\lambda kT\}-1}\right) = 0$$

whence

$$\frac{hc}{\lambda_{\max} kT}\left(1-\exp\{-hc/\lambda_{\max} kT\}\right)^{-1} - 5 = 0$$

An analytical solution of this equation can be obtained in terms of the Lambert W function; we merely quote the result,

$$\frac{\lambda_{\max}}{\mu\mathrm{m}} = \frac{2898}{T/\mathrm{K}}$$

We expect the Sun's output to peak around 500 nm (for $T_{\rm eff} = 5770$ K) – just where the human eye has peak sensitivity, for obvious evolutionary reasons.

1.7 Radiation Energy Density, U_ν

Consider some volume of space containing a given number of photons; the photons have energy, so we can discuss the density of radiant energy. From eqtn. (1.1), and referring to Fig. 1.1,

$$\mathrm{d}E_\nu = I_\nu(\theta)\,\mathrm{d}S\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega.$$

We can eliminate the time dependence[2] by noting that there is a single-valued correspondence[3] between time and distance for radiation. Defining a characteristic length $\ell = ct$, $\mathrm{d}t = \mathrm{d}\ell/c$, and

$$\begin{aligned}\mathrm{d}E_\nu &= I_\nu(\theta)\,\mathrm{d}S\frac{\mathrm{d}\ell}{c}\,\mathrm{d}\nu\,\mathrm{d}\Omega \\ &= \frac{I_\nu(\theta)}{c}\,\mathrm{d}V\,\mathrm{d}\nu\,\mathrm{d}\Omega \qquad (1.24)\end{aligned}$$

where the volume element $\mathrm{d}V = \mathrm{d}S\,\mathrm{d}\ell$. The mean radiation energy density per unit frequency per unit volume is then

$$\begin{aligned}U_\nu\,\mathrm{d}\nu &= \frac{1}{V}\int_V\int_\Omega \mathrm{d}E_\nu \\ &= \frac{1}{c}\int_\Omega I_\nu\,\mathrm{d}\nu\,\mathrm{d}\Omega\end{aligned}$$

[2] Assuming that no time dependence exists; that is, that for every photon leaving some volume of space, a compensating photon enters. This is an excellent approximation under many circumstances.

[3] Well, *nearly* single-valued; the speed at which radiation propagates actually depends on the refractive index of the medium through which it moves – e.g., the speed of light in water is only $^{3c}/_{4}$.

whence

$$U_\nu = \frac{1}{c}\int I_\nu \,\mathrm{d}\Omega$$
$$= \frac{4\pi}{c} J_\nu \qquad [\mathrm{J\ m^{-3}\ Hz^{-1}}] \qquad [\text{from eqtn. (1.2): } J_\nu = {}^1\!/\!{}_{4\pi}\int I_\nu \,\mathrm{d}\Omega] \tag{1.25}$$

Again, this is explicitly frequency dependent; the *total* energy density is obtained by integrating over frequency:

$$U = \int_0^\infty U_\nu \,\mathrm{d}\nu.$$

For black-body radiation, $J_\nu (= I_\nu) = B_\nu$, and

$$U = \int_0^\infty \frac{4\pi}{c} B_\nu \,\mathrm{d}\nu$$

but $\int \pi B_\nu = \sigma T^4$ (eqtn. (1.21)) so

$$U = \frac{4\sigma}{c} T^4 \equiv a T^4$$
$$= 7.55 \times 10^{-16}\, T^4 \qquad \mathrm{J\ m^{-3}} \tag{1.26}$$

where T is in kelvin, σ is the Stefan-Boltzmann constant and a is the 'radiation constant'. Note that the energy density of black-body radiation is a *fixed quantity* (for a given temperature).

For a given form of spectrum, the *energy* density in radiation must correspond to a specific *number* density of photons:

$$N_{\mathrm{photon}} = \int_0^\infty \frac{U_\nu}{h\nu} \,\mathrm{d}\nu.$$

For the particular case of a black-body spectrum,

$$N_{\mathrm{photon}} \simeq 2 \times 10^7\, T^3 \qquad \text{photons m}^{-3}. \tag{1.27}$$

Dividing eqtn. (1.26) by (1.27) gives the mean energy per photon for black-body radiation,

$$\overline{h\nu} = 3.78 \times 10^{-23} T = 2.74 kT \tag{1.28}$$

(although there is, of course, a broad spread in energies of individual photons).

1.8 Radiation Pressure

A photon carries momentum $E/c\,(= h\nu/c)$.[4] Momentum flux (momentum per unit time, per unit area) is a *pressure*.[5] If photons encounter a surface at some angle θ to the normal, the component of momentum perpendicular to the surface per unit time per unit area is that pressure,

$$\mathrm{d}P_\nu = \frac{\mathrm{d}E_\nu}{c} \times \cos\theta \frac{1}{\mathrm{d}t\,\mathrm{d}A\,\mathrm{d}\nu}$$

(where we have chosen to express the photon pressure 'per unit frequency'); but the specific intensity is

$$I_\nu = \frac{\mathrm{d}E_\nu}{\mathrm{d}A\cos\theta\,\mathrm{d}\Omega\,\mathrm{d}\nu\,\mathrm{d}t}, \tag{1.1}$$

whence

$$\mathrm{d}P_\nu = \frac{I_\nu}{c}\cos^2\theta\,\mathrm{d}\Omega$$

i.e.,

$$P_\nu = \frac{1}{c}\int I_\nu \mu^2\,\mathrm{d}\Omega \qquad [\mathrm{J\ m^{-3}\ Hz^{-1}} \equiv \mathrm{Pa\ Hz^{-1}}] \tag{1.29}$$

We know that

$$\int \mathrm{d}\Omega = \int_0^{2\pi}\int_{-1}^{+1} \mathrm{d}\mu\,\mathrm{d}\phi \tag{1.4}$$

so

$$P_\nu = \frac{2\pi}{c}\int_{-1}^{+1} I_\nu \mu^2\,\mathrm{d}\mu;$$

however, the K integral is

$$K_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\mu)\mu^2\,\mathrm{d}\mu \tag{1.14}$$

(from Section 1.4), hence

$$P_\nu = \frac{4\pi}{c}K_\nu \tag{1.30}$$

[4]Classically, momentum is mass times velocity. From $E = mc^2 = h\nu$, the photon rest mass is $h\nu/c^2$, and its velocity is c, hence momentum is $h\nu/c$.

[5]Dimensional arguments show this to be true; in the SI system, momentum has units of kg m s^{-1}, and momentum flux has units of kg m s^{-1}, m^{-2}, s^{-1}; i.e., kg m^{-1} s^{-2}, = N m^{-2} = Pa – the units of pressure. Pressure in turn is force per unit area (where force is measured in Newtons, = J m^{-1} = kg m s^{-2}).

For an *isotropic* radiation field $K_\nu = {}^1\!/\!{}_3 I_\nu = {}^1\!/\!{}_3 J_\nu$ (eqtn. (1.15)), and so

$$P_\nu = \frac{4\pi}{3c} I_\nu = \frac{4\pi}{3c} J_\nu.$$

In this isotropic case we also have

$$U_\nu = \frac{4\pi}{c} J_\nu \quad = \frac{4\pi}{c} I_\nu$$

(eqtn. (1.25)) so – for an isotropic radiation field – the radiation pressure is

$$P_\nu = \frac{1}{3} U_\nu$$

or, integrating over frequency (using $\int U_\nu \, \mathrm{d}\nu = aT^4 = {}^{4\sigma}\!/\!{}_c T^4$; eqtn. (1.26)),

$$P_\mathrm{R} = \frac{1}{3} aT^4 = \frac{4\sigma}{3c} T^4 \qquad [\mathrm{J\ m^{-3} \equiv N\ m^{-2} \equiv Pa}]. \tag{1.31}$$

In that equation (1.31) expresses the relationship between pressure and temperature, it is the equation of state for radiation.

Note that in the isotropic case, P_ν (or P_R) is a scalar quantity – it has magnitude but not direction (like air pressure, locally, on Earth). For an *anisotropic* radiation field, the radiation pressure has a direction (normally outwards from a star), and is a vector quantity. (This *directed* pressure, or force per unit area, becomes important in luminous stars, where the force becomes significant compared to gravity; Section 11.9.)

Section 2

The interaction of radiation with matter

As a beam of radiation traverses astrophysical material (such as a stellar interior, a stellar atmosphere, or interstellar space), energy can be added or subtracted – the process of 'radiative transfer'. A large number of detailed physical processes can contribute to these changes in intensity, and we will consider some of these processes in subsequent sections. First, though, we concentrate on general principles.

2.1 Emission: increasing intensity

A common astrophysical[1] definition of the (monochromatic) emissivity is the energy generated per unit volume,[2] per unit time, per unit frequency, per unit solid angle:

$$j_\nu = \frac{\mathrm{d}E_\nu}{\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega} \quad [\mathrm{J\ m^{-3}\ s^{-1}\ Hz^{-1}\ sr^{-1}}]; \tag{2.1}$$

If an element of distance along a line (e.g., the line of sight) is $\mathrm{d}s$, then the change in specific intensity along that element resulting from the emissivity of a volume of material of unit cross-sectional area is

$$\mathrm{d}I_\nu = +j_\nu(s)\,\mathrm{d}s \tag{2.2}$$

[1] Other definitions of 'emissivity' occur in physics.
[2] The emissivity can also be defined per unit mass (or, in principle, per particle).

2.2 Extinction: decreasing intensity

'Extinction' is a general term for the removal of light from a beam. Two different classes of process contribute to the extinction: absorption and scattering. Absorption (sometimes called 'true absorption') results in the destruction of photons; scattering merely involves redirecting photons in some new direction. For a beam directed towards the observer, scattering still has the effect of diminishing the recorded signal, so the two types of process can be treated together for the present purposes.

The amount of intensity removed from a beam by extinction in (say) a gas cloud must depend on

- The initial strength of the beam (the more light there is, the more you can remove)
- The number of particles (absorbers)
- The microphysics of the particles – specifically, how likely they are to absorb (or scatter) an incident photon. This microphysics is characterized by an effective cross-section per particle presented to the radiation.

By analogy with eqtn. (2.2), we can write the change in intensity along length $\mathrm{d}s$ as

$$\mathrm{d}I_\nu = -a_\nu n I_\nu \,\mathrm{d}s \tag{2.3}$$

for a number density of n extinguishing particles per unit volume, with a_ν the 'extinction coefficient', or *cross-section* (in units of area) per particle.

2.3 Opacity

In astrophysical applications, it is customary to combine the cross-section per particle (with dimensions of area) and the number of particles into either the extinction per unit mass, or the extinction per unit volume. In the former case we can set

$$a_\nu n \equiv \kappa_\nu \rho$$

and thus write eqtn. (2.3) as

$$\mathrm{d}I_\nu = -\kappa_\nu \rho(s) I_\nu \,\mathrm{d}s$$

for mass density ρ, where κ_ν is the (monochromatic) mass extinction coefficient or, more usually, the *opacity* per unit mass (dimensions of area per unit mass; SI units of $\mathrm{m}^2\ \mathrm{kg}^{-1}$).

For opacity per unit volume we have

$$a_\nu n \equiv k_\nu$$

whence

$$\mathrm{d}I_\nu = -k_\nu I_\nu \, \mathrm{d}s.$$

The volume opacity k_ν has dimensions of area per unit volume, or SI units of m^{-1}. It has a straightforward and useful physical interpretation; the mean free path for a photon moving through a medium with volume opacity k_ν is

$$\ell_\nu \equiv {}^{1}\!/_{k_\nu}. \tag{2.4}$$

[In the literature, κ is often used generically to indicate opacity, regardless of whether 'per unit mass' or 'per unit volume', and the sense has to be inferred from the context. (You can always do this by looking at the dimensions involved.)]

2.3.1 Optical depth

We can often calculate, but rarely measure, opacity as a function of position along a given path. Observationally, often all that is accessible is the cumulative effect of the opacity integrated along the line of sight; this is quantified by the *optical depth*,

$$\tau_\nu = \int_0^D k_\nu(s) \, \mathrm{d}s \qquad = \int_0^D \kappa_\nu \, \rho(s) \, \mathrm{d}s \qquad = \int_0^D a_\nu \, n(s) \, \mathrm{d}s \tag{2.5}$$

over distance D.

2.3.2 Opacity sources

At the atomic level, the processes which contribute to opacity are:

- bound-bound absorption (photoexcitation – line process);
- bound-free absorption (photoionization – continuum process);
- free-free absorption (continuum process); and
- scattering (continuum process).

Absorption process can be thought of as the destruction of photons (through conversion into other forms of energy, whether radiative or kinetic).

Scattering is the process of photon absorption followed by prompt re-emission through the inverse process. For example, resonance-line scattering is photo-excitation from the ground state to an excited state, followed quickly by radiative decay. Continuum scattering processes include electron scattering and Rayleigh scattering.

Under most circumstances, scattering involves re-emission of a photon with virtually the same energy (in the rest frame of the scatterer), but in a new direction.[3]

Calculation of opacities is a major task, but at the highest temperatures ($T \gtrsim 10^7$ K) elements are usually almost fully ionized, so free-free and electron-scattering opacities dominate. Under these circumstances, $\kappa \simeq$ constant. Otherwise, a parameterization of the form

$$\kappa = \kappa_0 \rho^a T^b \tag{2.6}$$

is convenient for analytical or illustrative work.

[3]In Compton scattering, energy is transferred from a high-energy photon to the scattering electron (or vice versa for inverse Compton scattering). These processes are important at X-ray and γ-ray energies; at lower energies, classical Thomson scattering dominates. For our purposes, 'electron scattering' can be regarded as synonymous with Thomson scattering.

Rate coefficients and rate equations (reference/revision only)

Before proceeding to consider specific astrophysical environments, we review the coefficients relating to *bound-bound* (line) transitions. Bound-free (ionization) process will be considered in sections 5 (photoionization) and 10.2 (collisional ionization).

Einstein (radiative) coefficients

Einstein (1916) proposed that there are three purely radiative processes which may be involved in the formation of a spectral line: induced emission, induced absorption, and spontaneous emission, each characterized by a coefficient reflecting the probability of a particular process.

[1] A_{ji} (s^{-1}): the Einstein coefficient, or transition probability, for spontaneous decay from an upper state j to a lower state i, with the emission of a photon (radiative decay); the time taken for an electron in state j to spontaneously decay to state i is $1/A_{ji}$ on average

If n_j is the number density of atoms in state j then the change in the number density of atoms in that state per unit time due to spontaneous emission will be

$$\frac{\mathrm{d}n_j}{\mathrm{d}t} = -\sum_{i<j} A_{ji} n_j$$

while level i is populated according to

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = +\sum_{j>i} A_{ji} n_j$$

[2] B_{ij} (s^{-1} J^{-1} m^2 sr): the Einstein coefficient for radiative excitation from a lower state i to an upper state j, with the absorption of a photon.

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = -\sum_{j>i} B_{ij} n_i I_\nu,$$

$$\frac{\mathrm{d}n_j}{\mathrm{d}t} = +\sum_{i<j} B_{ij} n_i I_\nu$$

[3] B_{ji} (s^{-1} J^{-1} m^2 sr): the Einstein coefficient for radiatively induced de-excitation from an upper state to a lower state.

$$\frac{\mathrm{d}n_j}{\mathrm{d}t} = -\sum_{i<j} B_{ji} n_j I_\nu,$$

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = +\sum_{j>i} B_{ji} n_j I_\nu$$

where I_ν is the specific intensity at the frequency ν corresponding to E_{ij}, the energy difference between excitation states.

For reference, we state, without proof, the relationships between these coefficients:

$$A_{ji} = \frac{2h\nu^3}{c^2} B_{ji};$$

$$B_{ij} g_i = B_{ji} g_j$$

where g_i is the statistical weight of level i.

In astronomy, it is common to work not with the Einstein A coefficient, but with the absorption oscillator strength f_{ij}, where

$$A_{ji} = \frac{8\pi^2 e^2 \nu^2}{m_e c^3} \frac{g_i}{g_j} f_{ij}$$

and f_{ij} is related to the absorption cross-section by

$$a_{ij} \equiv \int a_\nu \, \mathrm{d}\nu = \frac{\pi e^2}{m_\mathrm{e} c} f_{ij}.$$

Because of the relationships between the Einstein coefficients, we also have

$$B_{ij} = \frac{4\pi^2 e^2}{m_e h \nu c} f_{ij},$$

$$B_{ji} = \frac{4\pi^2 e^2}{m_e h \nu c} \frac{g_i}{g_j} f_{ij}$$

Collisional coefficients

For collisional processes we have analogous coefficients:

[4] C_{ji} ($\mathrm{m}^3\ \mathrm{s}^{-1}$): the coefficient for collisional de-excitation from an upper state to a lower state.

$$\frac{\mathrm{d}n_j}{\mathrm{d}t} = -\sum_{j>i} C_{ji} n_j n\mathrm{e},$$

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = +\sum_{i<j} C_{ji} n_j n\mathrm{e}$$

(for excitation by electron collisions)

[5] C_{ij} ($\mathrm{m}^3\ \mathrm{s}^{-1}$): the coefficient for collisional excitation from a lower state to an upper state.

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = -\sum_{j>i} C_{ij} n_i n\mathrm{e},$$

$$\frac{\mathrm{d}n_j}{\mathrm{d}t} = +\sum_{i<j} C_{ij} n_i n\mathrm{e}$$

These coefficients are related through

$$\frac{C_{ij}}{C_{ji}} = \frac{g_j}{g_i} \exp\left\{-\frac{h\nu}{kT_\mathrm{ex}}\right\}$$

for excitation temperature T_ex.

The rate coefficient has a Boltzmann-like dependence on the kinetic temperature

$$\begin{aligned} C_{ij}(T_\mathrm{k}) &= \left(\frac{2\pi}{T_\mathrm{k}}\right)^{1/2} \frac{h^2}{4\pi^2 m_\mathrm{e}^{3/2}} \frac{\Omega(ij)}{g_i} \exp\left\{\frac{-\Delta E_{ij}}{kT_\mathrm{k}}\right\} \\ &\propto \frac{1}{\sqrt{T_\mathrm{e}}} \exp\left\{\frac{-\Delta E_{ij}}{kT_\mathrm{k}}\right\} \qquad [\mathrm{m}^3\ \mathrm{s}^{-1}] \end{aligned} \tag{2.7}$$

where $\Omega(1,2)$ is the so-called ‘collision strength’.

Statistical Equilibrium

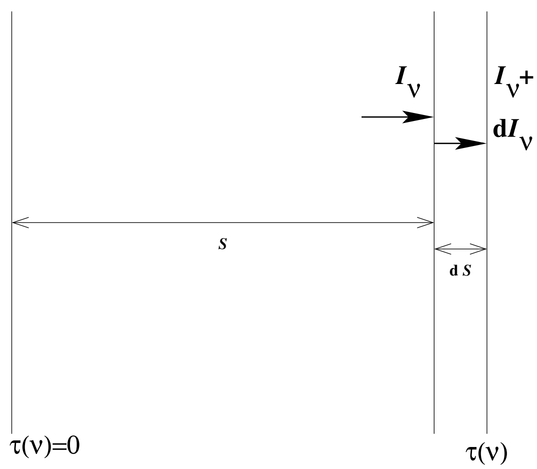
Overall, for any ensemble of atoms in equilibrium, the number of de-excitations from any given excitation state must equal the number of excitations into that state – the principle of *statistical equilibrium.* That is,

$$\sum_{j>i} B_{ij} n_i I_\nu + \sum_{j\neq i} C_{ij} n_i n\mathrm{e} = \sum_{j>i} A_{ji} n_j + \sum_{j>i} B_{ji} n_j I_\nu + \sum_{j\neq i} C_{ji} n_j n\mathrm{e} \tag{2.8}$$

Section 3

Radiative transfer

3.1 Radiative transfer along a ray



Consider a beam of radiation from a distant point source (e.g., an unresolved star), passing through some intervening material (e.g., interstellar gas). The intensity change as the radiation traverses the element of gas of thickness ds is the intensity added, less the intensity taken away (per unit frequency, per unit time, per unit solid angle):

$$\begin{aligned} \mathrm{d}I_\nu \,\cancel{(\mathrm{d}A\,\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}t)} = &+ j_\nu \quad \mathrm{d}s\,\cancel{\mathrm{d}A\,\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}t} \\ &- k_\nu\, I_\nu\, \mathrm{d}s\,\cancel{\mathrm{d}A\,\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}t} \end{aligned}$$

i.e.,

$$\mathrm{d}I_\nu = (j_\nu - k_\nu\, I_\nu)\,\mathrm{d}s,$$

or

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - k_\nu I_\nu, \tag{3.1}$$

which is the basic form of *the Equation of Radiative Transfer.*

The ratio j_ν/k_ν is called the *Source Function*, S_ν. For systems in thermodynamic equilibrium j_ν and k_ν are related through the Kirchhoff relation,

$$j_\nu = k_\nu B_\nu(T),$$

and so in this case (though not in general) the source function is given by the Planck function

$$S_\nu = B_\nu$$

Equation (3.1) expresses the intensity of radiation as a function of position. In astrophysics, we often can't establish exactly where the absorbers are; for example, in the case of an absorbing interstellar gas cloud of given physical properties, the same absorption lines will appear in the spectrum of some background star, regardless of where the cloud is along the line of sight. It's therefore convenient to divide both sides of eqtn. 3.1 by k_ν; then using our definition of optical depth, eqtn. (2.5), gives a more useful formulation,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = S_\nu - I_\nu. \tag{3.2}$$

3.1.1 Solution 1: $j_\nu = 0$

We can find simple solutions for the equation of transfer under some circumstances. The very simplest case is that of absorption only (no emission; $j_\nu = 0$), which is appropriate for interstellar absorption lines (or headlights in fog); just by inspection, eqtn. (3.1) has the straightforward solution

$$I_\nu = I_\nu(0)\exp\{-\tau_\nu\}\,. \tag{3.3}$$

We see that an optical depth of 1 results in a reduction in intensity of a factor e^{-1} (i.e., a factor ~ 0.37).

3.1.2 Solution 2: $j_\nu \neq 0$

To obtain a more general solution to transfer along a line we begin by guessing that

$$I_\nu = \mathcal{F}\exp\{C_1\tau_\nu\} \tag{3.4}$$

where $\mathcal{F}$ is some function to be determined, and C_1 some constant; differentiating eqtn. 3.4,

$$\begin{aligned}\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} &= \exp\{C_1\tau_\nu\}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\tau_\nu} + \mathcal{F}C_1\exp\{C_1\tau_\nu\} \\ &= \exp\{C_1\tau_\nu\}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\tau_\nu} + C_1 I_\nu, \qquad\qquad = S_\nu - I_\nu \text{ (eqtn. 3.2).}\end{aligned}$$

Identifying like terms we see that $C_1 = -1$ and that

$$S_\nu = \exp\{-\tau_\nu\}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\tau_\nu},$$

i.e.,

$$\mathcal{F} = \int_0^{\tau_\nu} S_\nu \exp\{t_\nu\}\,\mathrm{d}t_\nu + C_2$$

where t is a dummy variable of integration and C_2 is some constant. Referring back to eqtn. (3.4), we now have

$$I_\nu(\tau_\nu) = \exp\{-\tau_\nu\}\int_0^{\tau_\nu} S_\nu(t_\nu)\exp\{t_\nu\}\ \mathrm{d}t_\nu\ +\ I_\nu(0)\exp\{-\tau_\nu\}$$

where the constant of integration is set by the boundary condition of zero extinction ($\tau_\nu = 0$). In the special case of S_ν independent of τ_ν we obtain

$$I_\nu = I_\nu(0)\exp\{-\tau_\nu\} + S_\nu\left(1 - \exp\{-\tau_\nu\}\right)$$

3.2 Radiative Transfer in Stellar Atmospheres

Having established the principles of the simple case of radiative transfer along a ray, we turn to more general circumstances, where we have to consider radiation coming not just from one direction, but from arbitrary directions. The problem is now three-dimensional in principle; we could treat it in cartesian (xyz) coördinates,[1] but because a major application is in spherical objects (stars!), it's customary to use spherical polar coördinates.

Again consider a beam of radiation travelling in direction s, at some angle θ to the radial direction in a stellar atmosphere (Fig. 3.1). If we neglect the curvature of the atmosphere (the 'plane parallel approximation') and any azimuthal dependence of the radiation field, then the intensity change along this particular ray is

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - k_\nu I_\nu, \tag{3.1}$$

as before.

We see from the figure that

$$\mathrm{d}r = \cos\theta\,\mathrm{d}s \equiv \mu\mathrm{d}s$$

[1]We could also treat the problem as time-dependent; but we won't ... A further complication that we won't consider is motion in the absorbing medium (which introduces a directional dependence in k_ν and j_ν); this directionality is important in stellar winds, for example.

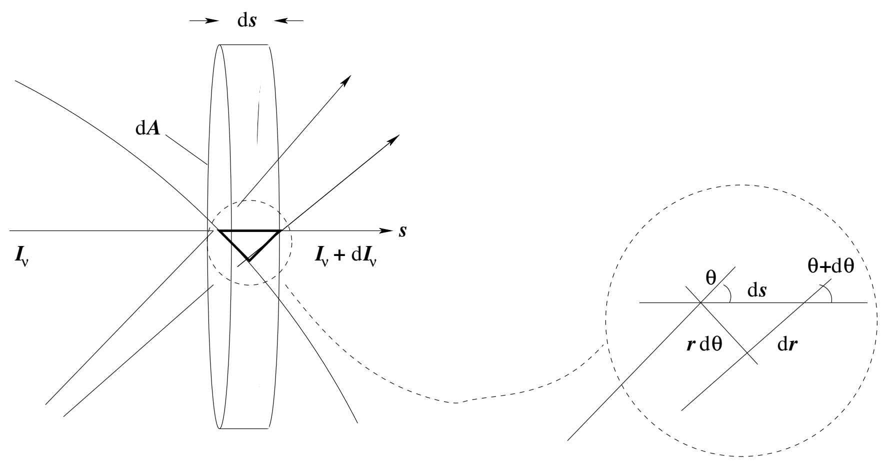


Figure 3.1: Geometry used in radiative-transfer discussion, section 3.2.

so the transfer in the radial direction is described by

$$\frac{\mu}{k_\nu}\frac{\mathrm{d}I_\nu}{\mathrm{d}r} = \frac{j_\nu}{k_\nu} - I_\nu, \tag{3.5}$$

(where we've divided through by k_ν); and since $j_\nu/k_\nu = S_\nu$ and $\mathrm{d}\tau_\nu = -k_\nu\,\mathrm{d}r$ (eqtn. 2.5, measuring distance in the radial direction, and introducing a minus because the sign convention in stellar-atmosphere work is such that optical depth *increases* with *decreasing* r) we have

$$\mu\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = S_\nu - I_\nu. \tag{3.6}$$

This is the standard formulation of the equation of transfer in plane-parallel stellar atmospheres.

For *arbitrary* geometry we have to consider the full three-dimensional characterization of the radiation field; that is

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \frac{\partial I_\nu}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}s}, \tag{3.7}$$

where r, θ, ϕ are our spherical polar coördinates. This is our most general formulation, but in the case of stellar atmospheres we can often neglect the ϕ dependence; and we rewrite the θ term by noting not only that

$$\mathrm{d}r = \cos\theta\,\mathrm{d}s \equiv \mu\mathrm{d}s$$

but also that

$$-r\,\mathrm{d}\theta = \sin\theta\,\mathrm{d}s.$$

(The origin of the minus sign may be clarified by reference to Fig. 3.1; for increasing s we have increasing r, but *decreasing* θ, so $r\,\mathrm{d}\theta$ is negative for positive $\mathrm{d}s$.)

Using these expressions in eqtn. (3.7) gives a two-dimensional form,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \frac{\partial I_\nu}{\partial r}\cos\theta - \frac{\partial I_\nu}{\partial \theta}\frac{\sin\theta}{r}$$

but this is also

$$= j_\nu - k_\nu I_\nu$$

so, dividing through by k_ν as usual,

$$\begin{aligned}\frac{\cos\theta}{k_\nu}\frac{\partial I_\nu}{\partial r} + \frac{\sin\theta}{k_\nu r}\frac{\partial I_\nu}{\partial\theta} &= \frac{j_\nu}{k_\nu} - I_\nu \\ &= S_\nu - I_\nu\end{aligned}$$

Once again, it's now useful to think in terms of the optical depth measured *radially inwards:*

$$\mathrm{d}\tau = -k_\nu\,\mathrm{d}r,$$

which gives us the customary form of the equation of radiative transfer for use in *extended* stellar atmospheres, for which the plane-parallel approximation fails:

$$\frac{\sin\theta}{\tau_\nu}\frac{\partial I_\nu}{\partial\theta} - \mu\frac{\partial I_\nu}{\partial\tau_\nu} = S_\nu - I_\nu. \tag{3.8}$$

We recover our previous, plane-parallel, result if the atmosphere is very thin compared to the stellar radius. In this case, the surface curvature shown in Fig. 3.1 becomes negligible, and $\mathrm{d}\theta$ tends to zero. Equation (3.8) then simplifies to

$$\mu\frac{\partial I_\nu}{\partial\tau_\nu} = I_\nu - S_\nu, \tag{3.6}$$

which is our previous formulation of the equation of radiative transfer in plane-parallel stellar atmospheres.

3.3 Energy transport in stellar interiors

3.3.1 Radiative transfer

In optically thick environments – in particular, stellar interiors – radiation is often the most important transport mechanism,[2] but for large opacities the radiant energy doesn't flow directly outwards; instead, it *diffuses* slowly outwards.

The same general principles apply as led to eqtn. (3.6); there is no azimuthal dependence of the radiation field, and the photon mean free path is (very) short compared to the radius. Moreover, we can make some further simplifications. First, the radiation field can be treated as isotropic to a very good approximation. Secondly, the conditions appropriate to 'local thermodynamic equilibrium' (LTE; Sec. 10.1) apply, and the radiation field is very well approximated by black-body radiation.

[2] Convection can also be a significant means of energy transport under appropriate conditions, and is discussed in Section 3.3.3.

Box 3.1. It may not be immediately obvious that the radiation field in stellar interiors is, essentially, isotropic; after all, outside the energy-generating core, the full stellar luminosity is transmitted across any spherical surface of radius r. However, if this flux is small compared to the local mean intensity, then isotropy is justified.

The flux at an interior radius r (outside the energy-generating core) must equal the flux at R (the surface); that is,

$$\pi F = \sigma T_{\rm eff}^4 \frac{R^2}{r^2}$$

while the mean intensity is

$$J_\nu(r) \simeq B_\nu(T(r)) = \sigma T^4(r).$$

Their ratio is

$$\frac{F}{J} = \left(\frac{T_{\rm eff}}{T(r)}\right)^4 \left(\frac{R}{r}\right)^2.$$

Temperature rises rapidly below the surface of stars, so this ratio is always small; for example, in the Sun, $T(r) \simeq 3.85$ MK at $r = 0.9\mathrm{R}_\odot$, whence $F/J \simeq 10^{-11}$. That is, the radiation field is isotropic to better than 1 part in 10^{11}.

We recall that, in general, I_ν is direction-dependent; i.e., is $I_\nu(\theta, \phi)$ (although we have generally dropped the explicit dependence for economy of nomenclature). Multiplying eqtn. (3.6) by $\cos\theta$ and integrating over solid angle, using $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \mathrm{d}\mu\,\mathrm{d}\phi$, then

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\nu}\int_0^{2\pi}\int_{-1}^{+1} \mu^2 I_\nu(\mu,\phi)\,\mathrm{d}\mu\,\mathrm{d}\phi = \int_0^{2\pi}\int_{-1}^{+1} \mu I_\nu(\mu,\phi)\,\mathrm{d}\mu\,\mathrm{d}\phi - \int_0^{2\pi}\int_{-1}^{+1} \mu S_\nu(\mu,\phi)\,\mathrm{d}\mu\,\mathrm{d}\phi;$$

or, for axial symmetry,

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\nu}\int_{-1}^{+1} \mu^2 I_\nu(\mu)\,\mathrm{d}\mu = \int_{-1}^{+1} \mu I_\nu(\mu)\,\mathrm{d}\mu - \int_{-1}^{+1} \mu S_\nu(\mu)\,\mathrm{d}\mu.$$

Using eqtns. (1.14) and (1.9), respectively, for the first two terms, and supposing that the emissivity has no preferred direction (as is true to an excellent aproximation in stellar interiors; Box 3.1) so that the source function is isotropic (and so the final term is zero), we obtain

$$\frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu} = \frac{F_\nu}{4\pi}$$

or, from eqtn. (1.15),

$$\frac{1}{3}\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = \frac{F_\nu}{4\pi}.$$

In LTE we may set $I_\nu = B_\nu(T)$, the Planck function; and $\mathrm{d}\tau_\nu = -k_\nu\,\mathrm{d}r$ (where again the minus arises because the optical depth is measured inwards, and decreases with increasing r). Making these substitutions, and integrating over frequency,

$$\int_0^\infty F_\nu\,\mathrm{d}\nu = -\frac{4\pi}{3}\int_0^\infty \frac{1}{k_\nu}\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}r}\mathrm{d}\nu \tag{3.9}$$

To simplify this further, we introduce the *Rosseland mean opacity,* $\bar{k}_{\rm R}$, defined by

$$\frac{1}{\bar{k}_{\rm R}}\int_0^\infty \frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}\mathrm{d}\nu = \int_0^\infty \frac{1}{k_\nu}\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}\mathrm{d}\nu.$$

Recalling that

$$\int_0^\infty \pi B_\nu \, \mathrm{d}\nu = \sigma T^4 \tag{1.21}$$

we also have

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}\mathrm{d}\nu &= \frac{\mathrm{d}}{\mathrm{d}T}\int_0^\infty B_\nu(T)\mathrm{d}\nu \\ &= \frac{4\sigma T^3}{\pi}\end{aligned}$$

so that eqtn. 3.9 can be written as

$$\int_0^\infty F_\nu \, \mathrm{d}\nu = -\frac{4\pi}{3}\frac{1}{\bar{k}_\mathrm{R}}\frac{\mathrm{d}T}{\mathrm{d}r}\frac{acT^3}{\pi} \tag{3.10}$$

where a is the radiation constant, $4\sigma/c$.

The luminosity at some radius r is given by

$$L(r) = 4\pi r^2 \int_0^\infty F_\nu \, \mathrm{d}\nu$$

so, finally,

$$L(r) = -\frac{16\pi}{3}\frac{r^2}{\bar{k}_\mathrm{R}}\frac{\mathrm{d}T}{\mathrm{d}r}acT^3, \tag{3.11}$$

which is our adopted form of the equation of radiative energy transport.

Box 3.2. The radiative energy density is $U = aT^4$ (eqtn. 1.26), so that $\mathrm{d}U/\mathrm{d}T = 4aT^3$, and we can express eqtn. (3.10) as

$$\begin{aligned}F &= \int_0^\infty F_\nu \, \mathrm{d}\nu \\ &= -\frac{c}{3\bar{k}_\mathrm{R}}\frac{\mathrm{d}T}{\mathrm{d}r}\frac{\mathrm{d}U}{\mathrm{d}T} \\ &= -\frac{c}{3\bar{k}_\mathrm{R}}\frac{\mathrm{d}U}{\mathrm{d}r}\end{aligned}$$

This 'diffusion approximation' shows explicitly how the radiative flux relates to the energy gradient; the constant of proportionality, $c/3\bar{k}_\mathrm{R}$, is called the diffusion coefficient. The larger the opacity, the less the flux of radiative energy, as one might intuitively expect.

3.3.2 Convection in stellar interiors

Energy transport can take place through one of three standard physical processes: radiation, convection, or conduction. In the rarified conditions of interstellar space, radiation is the only significant mechanism; and gases are poor conductors, so conduction is generally negligible even in stellar interiors (though not in, e.g., neutron stars). In stellar interiors (and some stellar atmospheres) energy transport by convection can be very important.

Conditions for convection to occur

We can rearrange eqtn. (3.11) to find the temperature gradient where energy transport is radiative:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi}\frac{\overline{k}_{\mathrm{R}}}{r^2}\frac{L(r)}{acT^3},$$

If the energy flux isn't contained by the temperature gradient, we have to invoke another mechanism – convection – for energy transport. (Conduction is negligible in ordinary stars.) Under what circumstances will this arise?

Suppose that through some minor perturbation, an element (or cell, or blob, or bubble) of gas is displaced upwards within a star. It moves into surroundings at lower pressure, and if there is no energy exchange it will expand and cool adiabatically. This expansion will bring the system into pressure equilibrium (a process whose timescale is naturally set by the speed of sound and the linear scale of the perturbation), but not *necessarily* temperature equilibrium – the cell (which arose in deeper, hotter layers) may be hotter and less dense than its surroundings. If it is less dense, then simple buoyancy comes into play; the cell will continue to rise, and convective motion occurs.[3]

We can establish a condition for convection by conmsidering a discrete bubble of gas moving upwards within a star, from radius r to $r + \mathrm{d}r$. We suppose that the pressure and density of the ambient background and within the bubble are
(P_1, ρ_1), (P_2, ρ_2) and (P_1^*, ρ_1^*), (P_2^*, ρ_2^*),
respectively, at (r), $(r + \mathrm{d}r)$.

The condition for adiabatic expansion is that

$$PV^\gamma = \text{constant}$$

where $\gamma = C_P/C_V$, the ratio of specific heats at constant pressure and constant volume. (For a monatomic ideal gas, representative of stellar interiors, $\gamma = 5/3$.) Thus, for a blob of constant mass ($V \propto \rho^{-1}$),

$$\frac{P_1^*}{(\rho_1^*)^\gamma} = \frac{P_2^*}{(\rho_2^*)^\gamma}; \text{ i.e.,}$$

$$(\rho_2^*)^\gamma = \frac{P_2^*}{P_1^*}(\rho_1^*)^\gamma.$$

A displaced cell will continue to rise if $\rho_2^* < \rho_2$. However, from our discussion above, we suppose that $P_1^* = P_1$, $\rho_1^* = \rho_1$ initially; and that $P_2^* = P_2$ finally. Thus

[3]Another way of looking at this is that the entropy (per unit mass) of the blob is conserved, so the star is unstable if the ambient entropy per unit mass decreases outwards.

convection will occur if

$$(\rho_2^*)^\gamma = \frac{P_2}{P_1}(\rho_1)^\gamma < \rho_2.$$

Setting $P_2 = P_1 - \mathrm{d}P$, $\rho_2 = \rho_1 - \mathrm{d}\rho$, we obtain the condition

$$\rho_1 \left(1 - \frac{\mathrm{d}P}{P_1}\right)^{1/\gamma} < \rho_1 - \mathrm{d}\rho.$$

If $\mathrm{d}P \ll P_1$, then a binomial expansion gives us

$$\left(1 - \frac{\mathrm{d}P}{P}\right)^{1/\gamma} \simeq 1 - \frac{1}{\gamma}\frac{\mathrm{d}P}{P}$$

(where we have dropped the now superfluous subscript), and so

$$\begin{aligned} -\frac{\rho}{\gamma}\frac{\mathrm{d}P}{P} &< -\mathrm{d}\rho, \text{ or} \\ -\frac{1}{\gamma}\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r} &< -\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} \end{aligned} \tag{3.12}$$

However, the equation of state of the gas is $P \propto \rho T$, i.e.,

$$\begin{aligned} \frac{\mathrm{d}P}{P} &= \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}T}{T}, \text{ or} \\ \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r} &= \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}; \end{aligned}$$

thus, from eqtn. (3.12),

$$\begin{aligned} \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r} &< \frac{1}{\gamma}\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r} \\ &< \left(\frac{\gamma}{\gamma - 1}\right)\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}, \text{ or} \\ \frac{\mathrm{d}(\ln P))}{\mathrm{d}(\ln T)} &< \frac{\gamma}{\gamma - 1} \end{aligned} \tag{3.13}$$

for convection to occur.

Schwarzschild criterion

Start with adiabatic EOS

$$P\rho^{-\gamma} = \text{constant} \tag{3.14}$$

i.e.,

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln P} = \frac{1}{\gamma} \tag{3.15}$$

To rise, the cell density must decrease more rapidly than the ambient density; i.e.,

$$\frac{\mathrm{d}\ln\rho_c}{\mathrm{d}\ln P_c} > \frac{\mathrm{d}\ln\rho_a}{\mathrm{d}\ln P_a} \tag{3.16}$$

Gas law, $P = nkT = \rho kT/\mu$ gives

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln P} = 1 + \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln P} - \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} \tag{3.17}$$

whence the Schwarzschild criterion for convection,

$$\frac{\mathrm{d}\ln T_a}{\mathrm{d}\ln P_a} > 1 - \frac{1}{\gamma} + \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln P}. \tag{3.18}$$

3.3.3 Convective energy transport

Convection is a complex, hydrodynamic process. Although much progress is being made in numerical modelling of convection over short timescales, it's not feasible at present to model convection in detail in stellar-evolution codes, because of the vast disparities between convective and evolutionary timescales. Instead, we appeal to simple parameterizations of convection, of which mixing-length 'theory' is the most venerable, and the most widely applied.

We again consider an upwardly moving bubble of gas. As it rises, a temperature difference is established with the surrounding (cooler) gas, and in practice some energy loss to the surroundings must occur.

XXXWork in progress