Numerical integration of RT
in a simplest case

- Local Thermodynamical Equilibrium (LTE, all microprocesses are in detailed balance)
- Static (no time dependence)
- Simple geometry (e.g semi-infinit medium)
- One dimension

Main equations

Equation of radiative transfer:

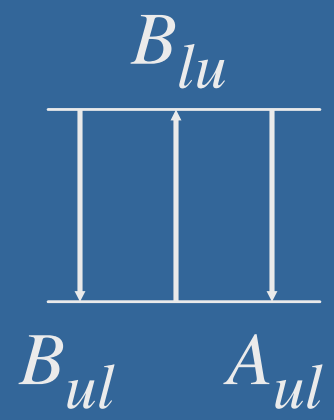
$$\frac{dI_\nu}{dx_\nu} = -k_\nu(x)\cdot\rho(x)\cdot I_\nu + k_\nu(x)\cdot\rho(x)\cdot S_\nu(x)$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

For the case of semi-infinit medium (e.g. stellar atmosphere) boundary condition is set deep (inside a star):

$$I_\nu(\tau_\infty) = B_\nu(T_\infty)$$

Einstein coefficients



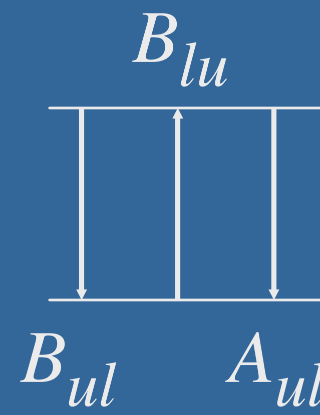
- A_{ul} - spontaneous de-excitation.
- B_{lu} - radiative excitation.
- B_{ul} - stimulated de-excitation.
- Einstein relations connect the probabilities:

$$\frac{B_{lu}}{B_{ul}} = \frac{g_u}{g_l}; \qquad \frac{A_{ul}}{B_{ul}} = \frac{2h\nu^3}{c^2}$$

and that *lu* transition rate match the *ul* rate:

$$n_l B_{lu} I_\nu = n_u A_{ul} + n_u B_{ul} I_\nu$$

Absorption/Emission



- Energy absorbed:

$$k_\nu^{bb}\rho \cdot I_\nu = \frac{h\nu}{4\pi} \cdot n_l \cdot B_{lu} \cdot \varphi(\nu - \nu_0) \cdot I_\nu$$

- Energy emitted:

$$j_\nu^{bb}\rho = \frac{h\nu}{4\pi} \cdot n_u \cdot B_{ul} \cdot \chi(\nu - \nu_0) \cdot I_\nu + \\ + \frac{h\nu}{4\pi} \cdot n_u \cdot A_{ul} \cdot \psi(\nu - \nu_0)$$

- Probability profiles are area normalized:

$$\int_0^\infty \varphi(\nu - \nu_0) d\nu = \int_0^\infty \psi(\nu - \nu_0) d\nu = \int_0^\infty \chi(\nu - \nu_0) d\nu = 1$$

Absorption/Emission

- Absorption coefficient expressed through Einstein probabilities:

$$k_\nu^{bb}\rho = \frac{h\nu}{4\pi}\left[n_l B_{lu}\varphi(\nu-\nu_0) - n_u B_{ul}\chi(\nu-\nu_0)\right] =$$

$$= \frac{h\nu}{4\pi} n_l B_{lu}\varphi(\nu-\nu_0)\left[1 - \frac{n_u g_l \chi(\nu-\nu_0)}{n_l g_u \varphi(\nu-\nu_0)}\right]$$

- Emission coefficient:

$$j_\nu^{bb}\rho = \frac{h\nu}{4\pi}\cdot n_u \cdot A_{ul}\cdot \psi(\nu-\nu_0)$$

- LTE (detailed balance for each frequency) means that probability profiles are the same:

$$\varphi(\nu-\nu_0) = \psi(\nu-\nu_0) = \chi(\nu-\nu_0)$$

Source function

- Source function:

$$S_\nu^{bb} = \frac{j_\nu^{bb}}{k_\nu^{bb}} = \frac{n_u A_{ul} \psi(\nu - \nu_0)}{\left[n_l B_{lu} \varphi(\nu - \nu_0) - n_u B_{ul} \chi(\nu - \nu_0) \right]}$$

- LTE (detailed balance for each frequency) means that probability profiles are the same:

$$\varphi(\nu - \nu_0) = \psi(\nu - \nu_0) = \chi(\nu - \nu_0)$$

- Level population in LTE is described by the Boltzmann distribution:

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-\frac{\Delta E}{\mathrm{k}T}} = \frac{g_u}{g_l} e^{-\frac{h\nu}{\mathrm{k}T}}$$

Source function and Absorption coefficient in LTE

- Source function in LTE:

$$S_\nu^{bb} = \frac{n_u A_{ul} \psi(\nu - \nu_0)}{\left[n_l B_{lu} \varphi(\nu - \nu_0) - n_u B_{ul} \chi(\nu - \nu_0)\right]} =$$

$$= \frac{n_u A_{ul}}{\left[n_l B_{lu} - n_u B_{ul}\right]} = \frac{A_{ul} / B_{ul}}{\left(n_l / n_u \cdot B_{lu} / B_{ul} - 1\right)} =$$

$$= \frac{2h\nu^3}{c^2} \cdot \frac{1}{\left(e^{h\nu/kT} - 1\right)} = B_\nu(T)$$

- Absorption coefficient in LTE:

$$k_\nu^{bb} \rho = \frac{h\nu}{4\pi} n_l B_{lu} \varphi(\nu - \nu_0) \cdot \left(1 - e^{-h\nu/kT}\right)$$

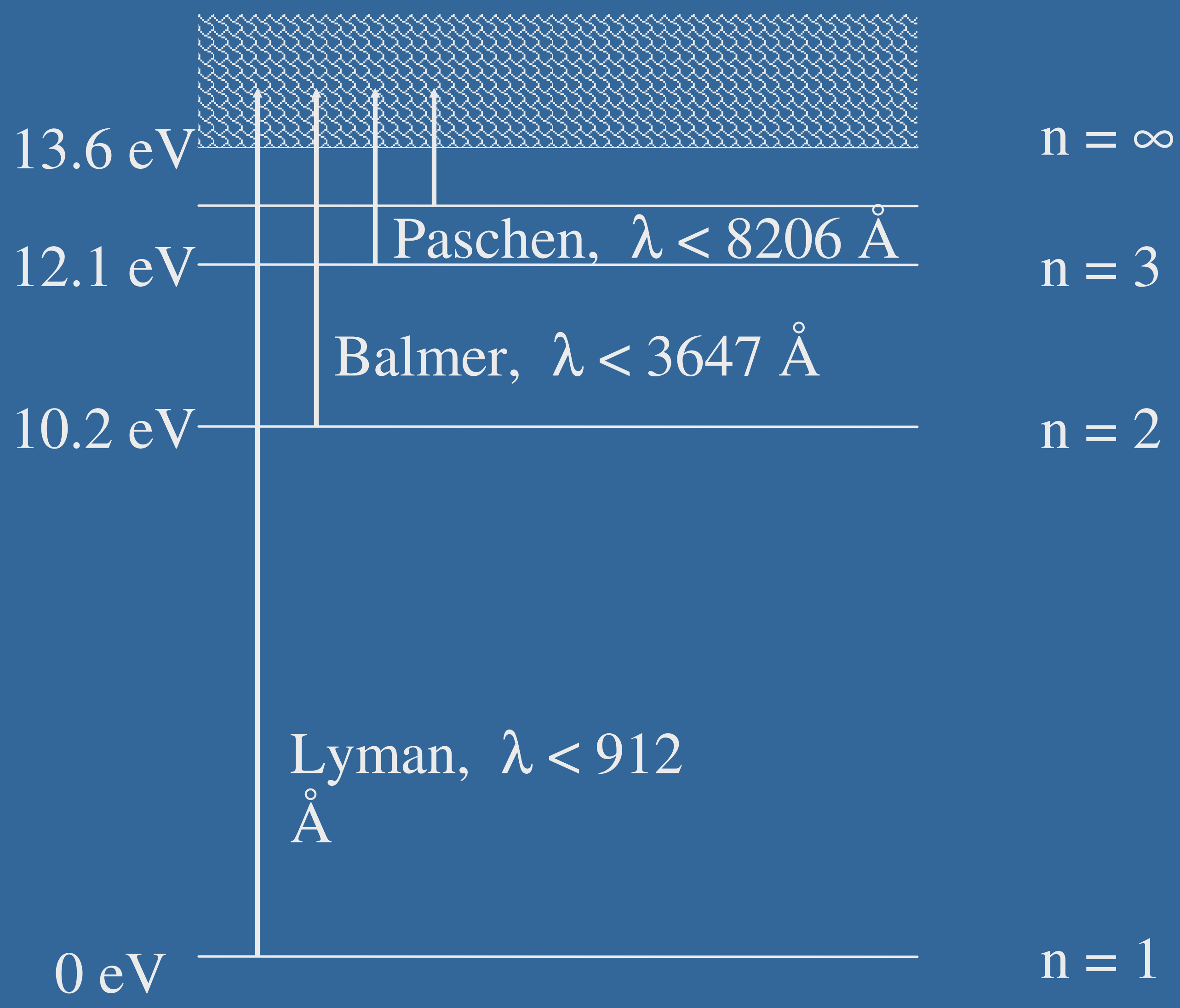
Continuous opacity

- Continuous opacity includes *b-f* (photoionization) and *f-f* transitions.
- Neutral Hydrogen is often a dominating source due to its abundance.
- For *b-f* transitions only photons with energy larger than the difference between the ionization energy and the energy of a bound level can be absorbed:

$$h\nu = h\mathrm{R}c/n^2 - m_\mathrm{e}\upsilon^2/2$$

This produces the absorption edges.

b-f transitions in Hydrogen



Total opacity in LTE

- *b-f* and *f-f* opacities are described by the same expression for opacity coefficient as for *b-b*. Just B_{ul} and the absorption profiles are different.
- The source function is still a Planck function.
- Total absorption:

$$k_\nu = k_\nu^{bb} + k_\nu^{bf} + k_\nu^{ff}$$

Line profile

- The last thing left is the absorption probability profile.
- Spectral lines are not delta-functions due to three effects:

Lorentz

  - Damping of radiation
  - Perturbation of atomic energy level system by neighboring particles

Gaussian

  - Doppler movements of absorbers/emitters

- The convolution the Lorentz and Doppler profile results in Voigt profile:

$$\mathrm{H}(a,v) = \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{-y^2}}{(v-y)^2+a^2}dy$$

Now... we know everything

- RT equation:

$$\frac{dI_\nu}{dx} = -k_\nu \cdot \rho \cdot I_\nu + k_\nu \cdot \rho \cdot S_\nu$$

- Boundary condition:

$$I_\nu(\tau_\infty) = B_\nu(T_\infty)$$

- Absorption coefficient:

$$k_\nu = k_\nu^{bb} + k_\nu^{bf} + k_\nu^{ff}$$

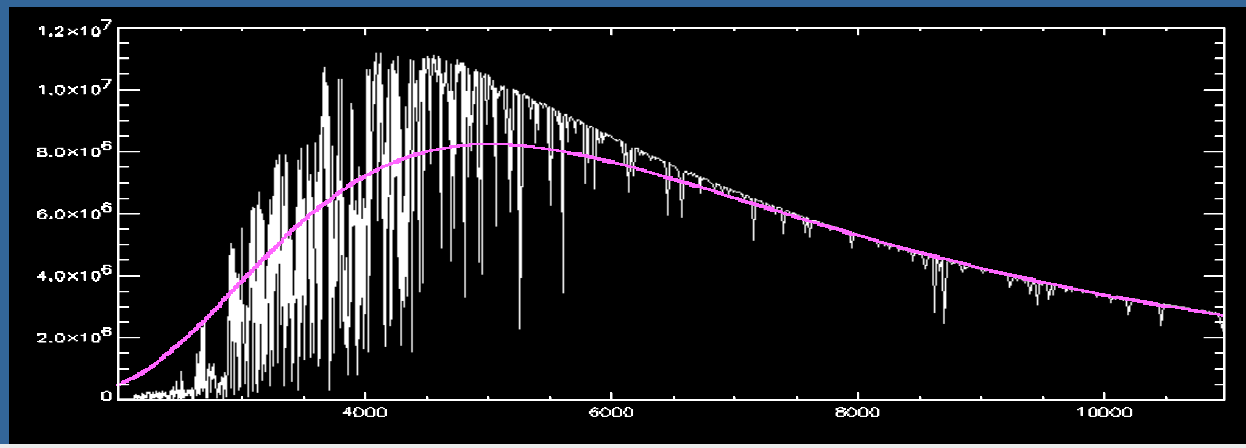
- Absorption profile:

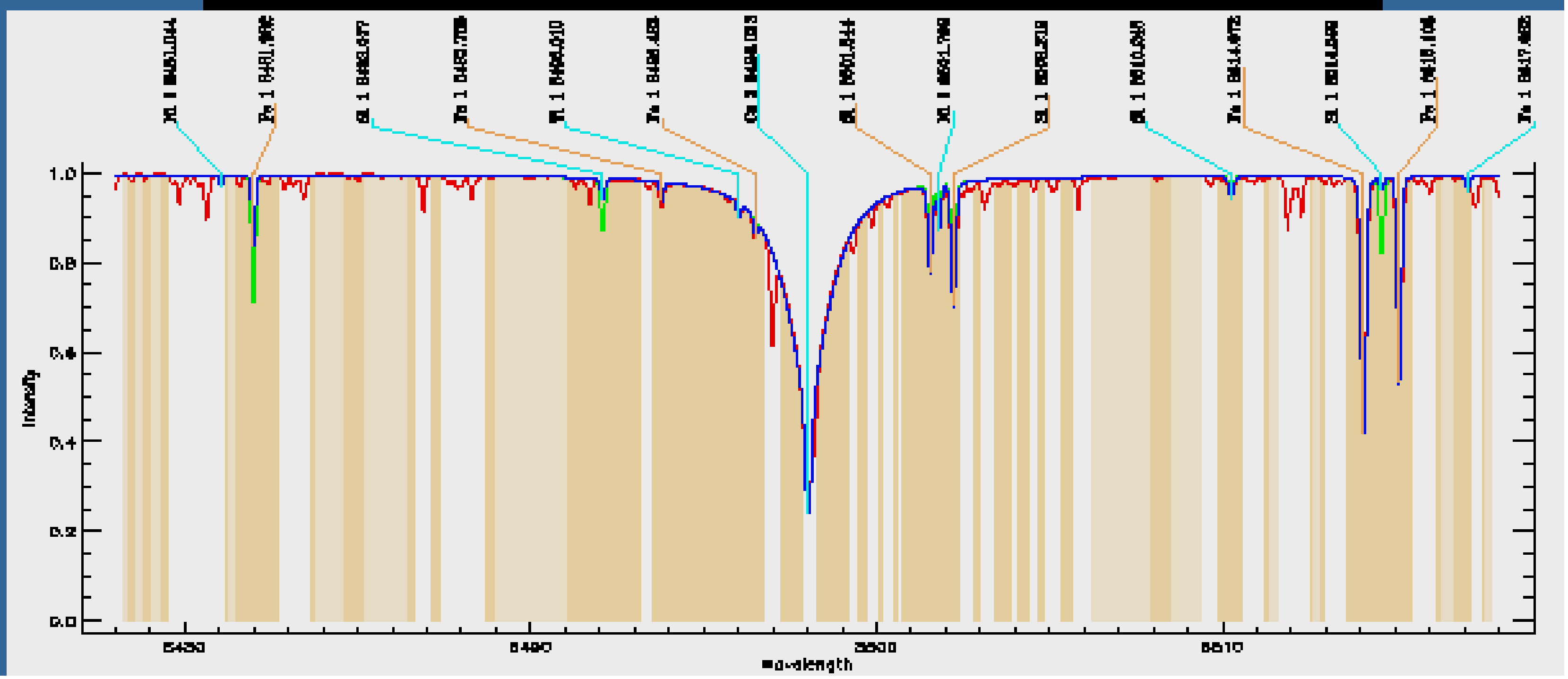
$$\mathrm{H}(a, v)$$

- Source function

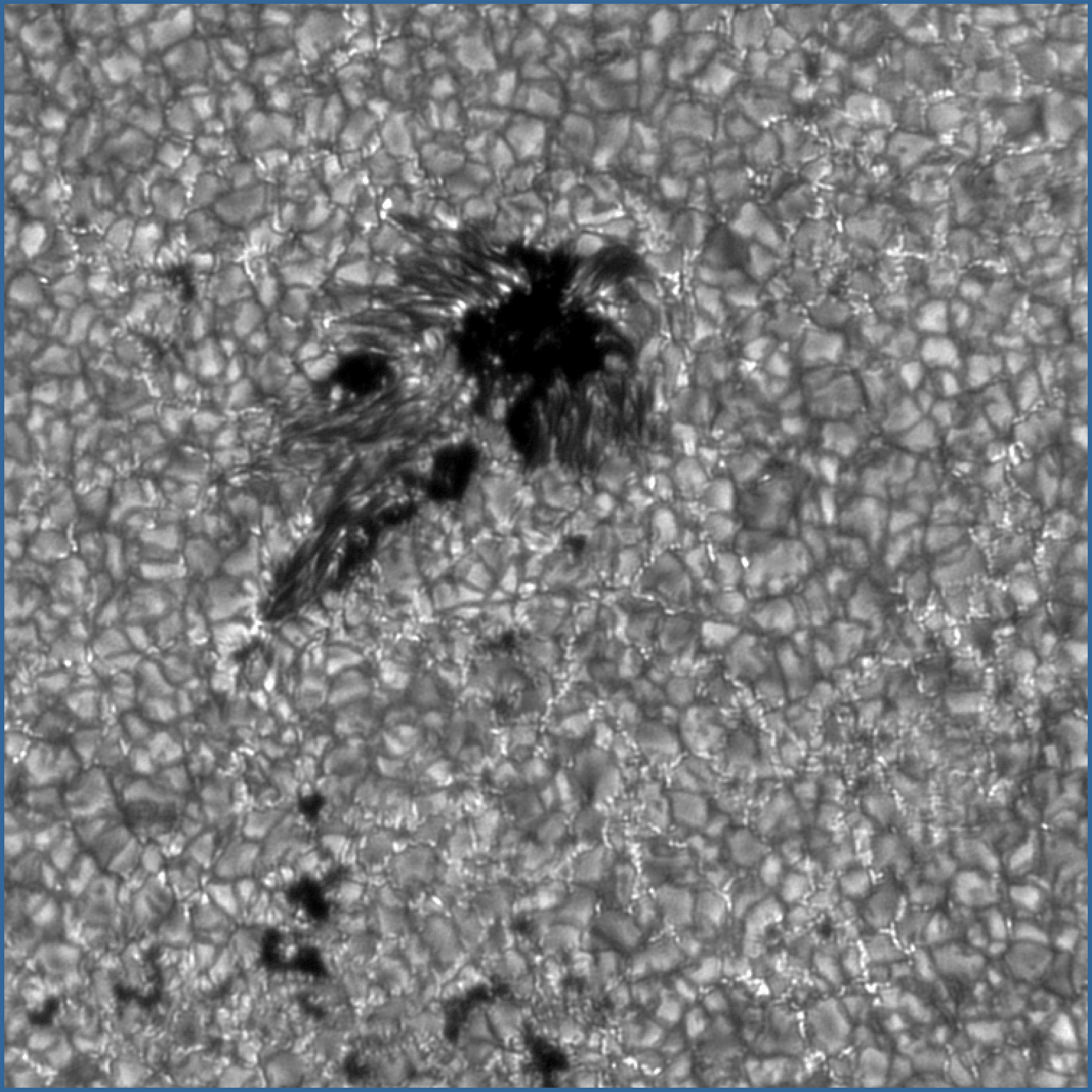
$$B_\nu(T)$$

How good is LTE for solving RT?





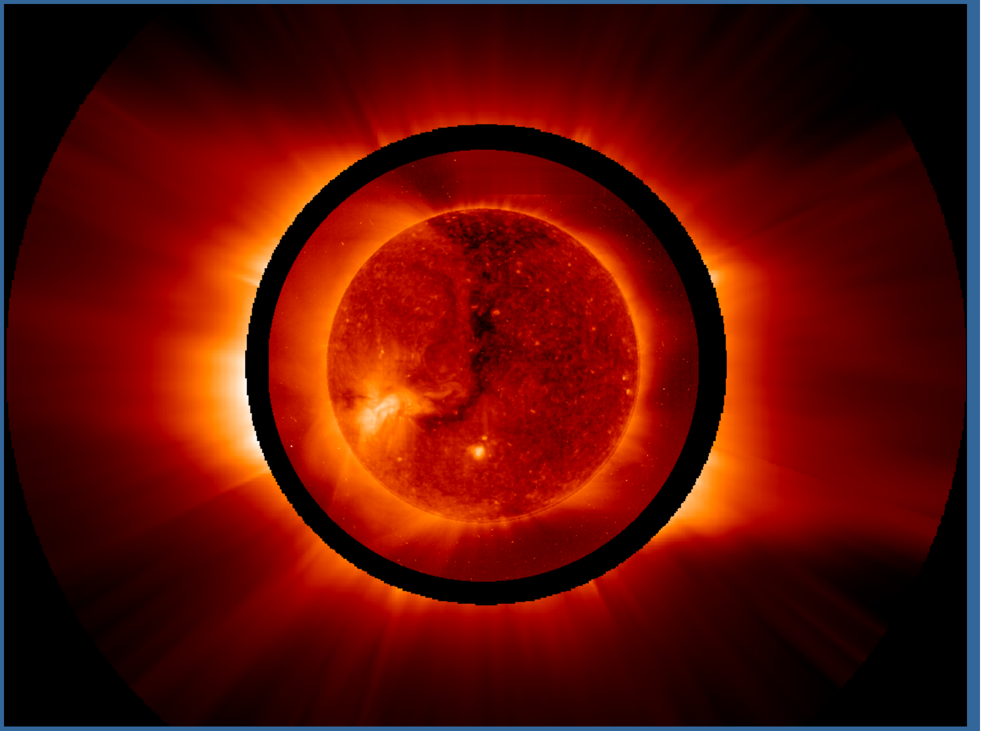
Stellar surfaces



Courtesy of Göran Scharmer/Swedish Solar Vacuum Telescope

Energy transport in stars

- Radiation
- Convection
- Particle ejection
- Waves

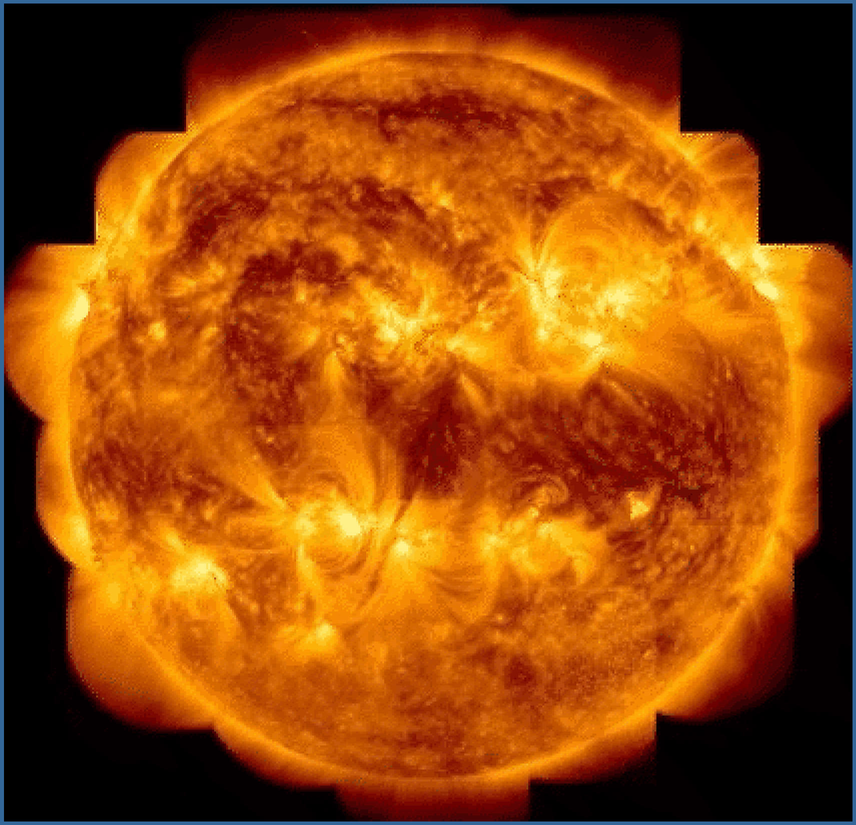


Courtesy of SOHO

Space above surface is not empty!

Solar corona

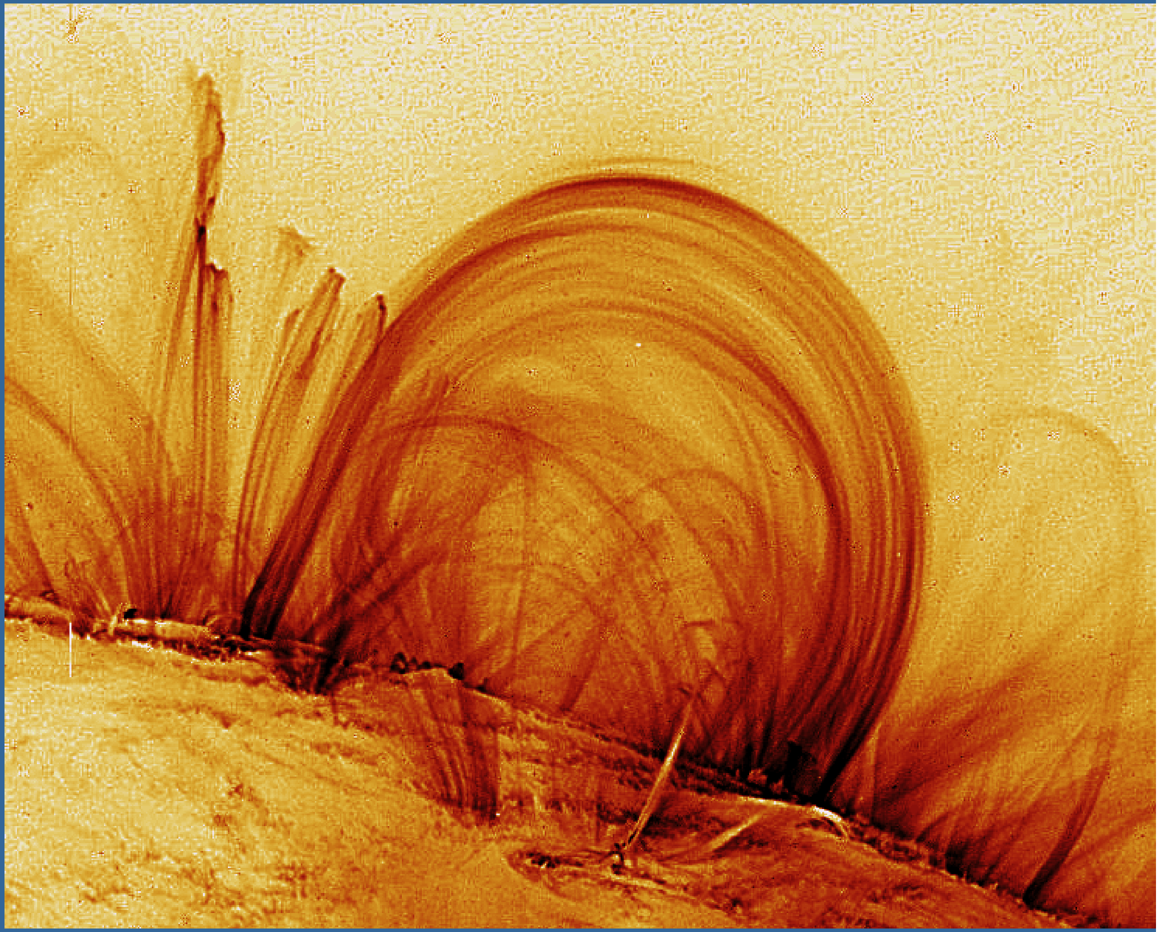
- Low density: $<10^7$ particle/cm^3
- Hot: $>10^6$ K
- Optically thin
- Temperature of radiation is very different from the kinetic temperature



Courtesy of SOHO

Coronal arcades on the Sun

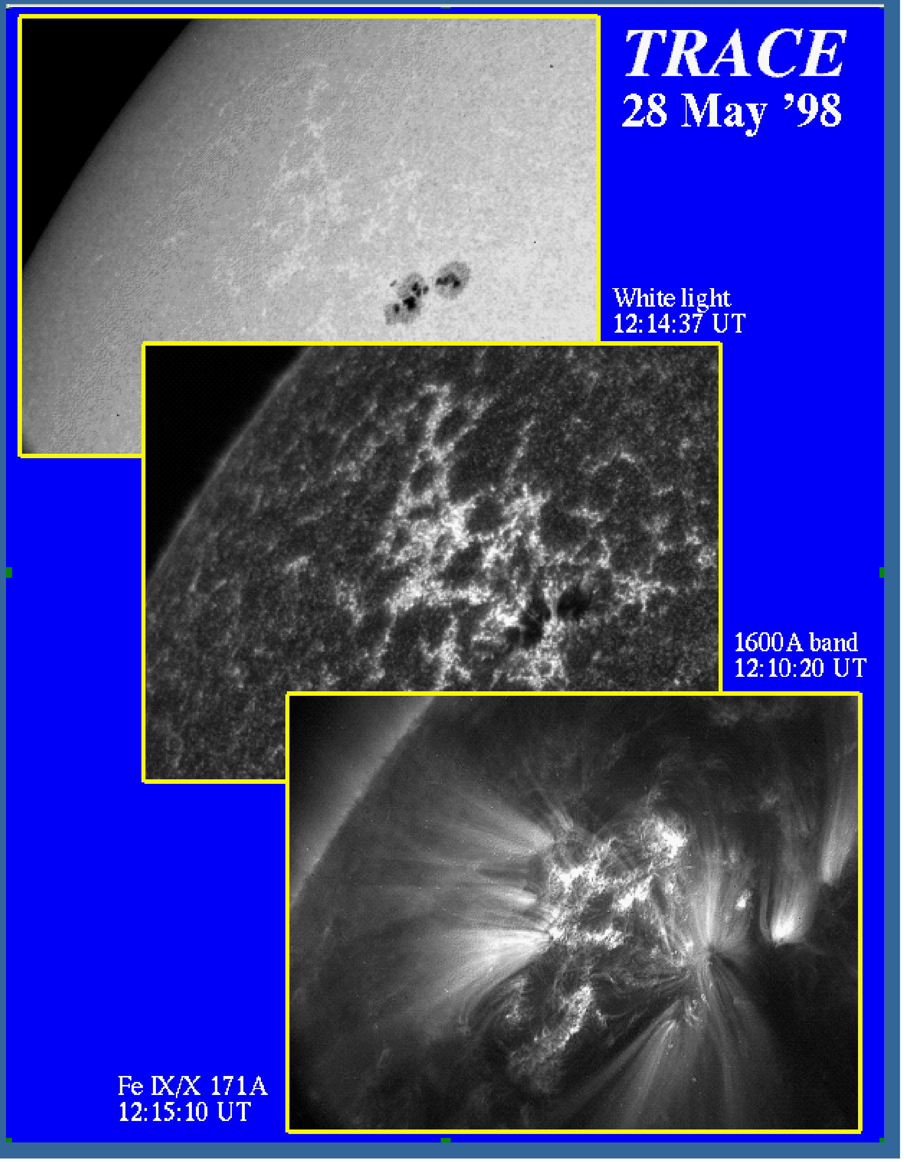
- Heating of the outer layers is part of the energy transport
- Magnetic fields play a major role
- Coherent motions at the photosphere layers dissipate in the corona making it hot



Courtesy of Karel Schrijver/TRACE

How can observe space above solar surface?

- At visual spectral range photosphere dominates the total flux
- UV lines allow to see chromospheric structures
- Going to X-ray is required to observe solar corona



Courtesy of Karel Schrijver/TRACE

Practical implementation

- Maxwellian velocity distribution:

$$\mathrm{P}(\upsilon_x) = \frac{1}{\sqrt{\pi}\cdot\overline{\upsilon}}\cdot e^{-\left(\frac{\upsilon_x}{\overline{\upsilon}}\right)^2} \; ; \quad \overline{\upsilon} = \sqrt{\frac{2kT}{m_{\mathrm{A}}}}$$

- Boltzmann level population

$$\frac{n_{\mathrm{u}}}{n_{\mathrm{l}}} = \frac{g_{\mathrm{u}}}{g_{\mathrm{l}}} e^{-\frac{\left(E_{\mathrm{u}}-E_{\mathrm{l}}\right)}{kT}}$$

- Saha ionization balance

$$\frac{n_{i+1}}{n_i} = \frac{1}{N_e}\cdot\frac{2\,\mathrm{U}_{i+1}}{\mathrm{U}_i}\cdot\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\cdot e^{-\frac{E_i}{kT}}$$

Solving RT

Simple minded approach: RK

$$\frac{dy}{dx} = f(x) \cdot y + g(x); \;\; y(x_0) = y_0$$

$$k_1 = f(x_i) \cdot y_i + g(x_i)$$

$$k_2 = f\left(x_i + \frac{h}{2}\right) \cdot \left(y_i + \frac{hk_1}{2}\right) + g\left(x_i + \frac{h}{2}\right)$$

$$k_3 = f\left(x_i + \frac{h}{2}\right) \cdot \left(y_i + \frac{hk_2}{2}\right) + g\left(x_i + \frac{h}{2}\right)$$

$$k_4 = f\left(x_i + h\right) \cdot \left(y_i + hk_3\right) + g\left(x_i + h\right)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

More clever RK. Previous example suffers from all problems inherent to RK, specially of we deal with complex medium where *f* and *g* have complex behavior. Instead one can solve RK analytically:

$$I_\nu(\tau''_\nu) = I_\nu(\tau'_\nu) \cdot e^{-\Delta\tau_\nu} + \int_{\tau'_\nu}^{\tau''_\nu} B_\nu(t) \cdot e^{-(\tau''_\nu - t)} dt$$

In particular, this is useful for a half-infinite medium where we can easily use Gauss quadratures for the integral:

$$I_\nu(0) = \int_0^\infty B_\nu(t) \cdot e^{-t} dt = \sum_{i=1}^{N} \omega_i \cdot B_\nu(\tau_{\nu,i})$$

The nodes and weights of Laguerre polynomials:

0.137793470540	3.0844115765E-01
0.729454549503	4.0111992915E-01
1.808342901740	2.1806828761E-01
3.401433697855	6.2087456098E-02
5.552496140064	9.5015169751E-03
8.330152746764	7.5300838858E-04
11.843785837900	2.8259233496E-05
16.279257831378	4.2493139849E-07

The only problem is that values of ***T*** are not known in $\tau_{\nu,i}$ We can find them solving ODE for optical depth:

$$\frac{dx}{d\tau_\nu} = \frac{1}{k_\nu(x)\cdot\rho(x)}, \qquad x\Big|_{\tau_\nu=0} = 0$$

Advantages: *simple boundary conditions and right hand side does not depend on unknown function.*

4th order Runge-Kutta for geometrical depth

$$\frac{dx}{d\tau_\nu} = \frac{1}{k_\nu(x)}; \qquad x_0 = 0$$

$$k_1 = \frac{1}{k_\nu(x_i)}; \qquad k_2 = \frac{1}{k_\nu\left(x_i + hk_1/2\right)}$$

$$k_3 = \frac{1}{k_\nu\left(x_i + hk_2/2\right)}; \quad k_4 = \frac{1}{k_\nu\left(x_i + hk_3\right)}$$

$$x_{i+1} = x_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

We integrate the equation for x from 0 to each of the $\tau_{\nu,i}$ consecutively. For each x_i we find the temperature and then intensity using Gauss quadrature.