Introduction to Numerical Hydrodynamics and Radiative Transfer

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From simple LTE radiative transfer in static medium
to multi-dimensional numerical schemes in dynamic environments

Goals

- *Get* a consistent picture of *radiative energy transport as part of hydrodynamics of complex media* using examples of typical astrophysical environment
- *Learn numerical methods and approximations describing hydrodynamics and radiative transfer*, understand the *advantages and limitations of different techniques*
- *Obtain* initial *experience in programming* and using state-of-the-art computers including *parallel machines*

Summary of the Course

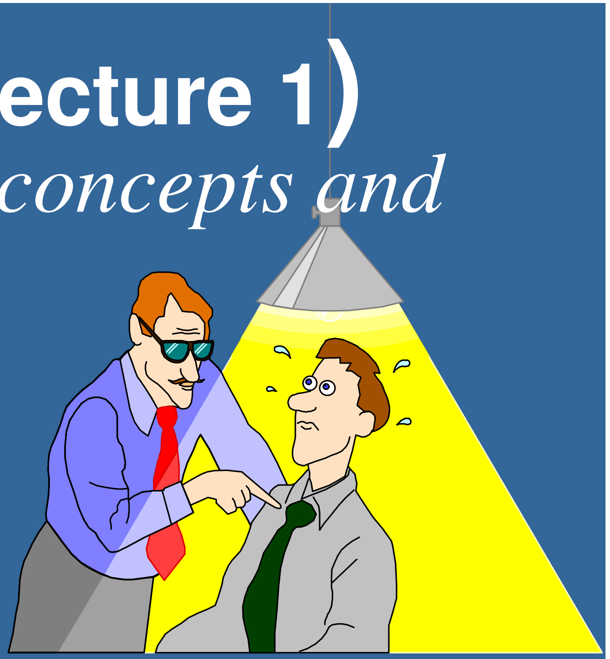
- <u>Structure:</u> The course will consists of 2 parts: lectures and two (only!) exercises.
- <u>Grading:</u> In order to complete the course students would have to attend most of the lectures, do the home work, and successfully complete the basic level exercises in RT and HD. This will result in 5 points. Additional 2 points will be awarded to the students who would complete the advanced level exercise.

Schedule

- We hope to finish all the lectures before mid January 2005
- 2 lectures a week: ***Mondays*** and ***Thursdays 10am-noon***.
- Logical sequence
  - Recall of RT and Numerical Methods
  - Numerical Models of Stellar Atmospheres $\rightarrow$ microphysics $\rightarrow$ NLTE calculations
  - Hydrodynamical equations $\rightarrow$ detailed description in 1D $\rightarrow$ generalization to 3D
  - Combining of HD and RT in one model

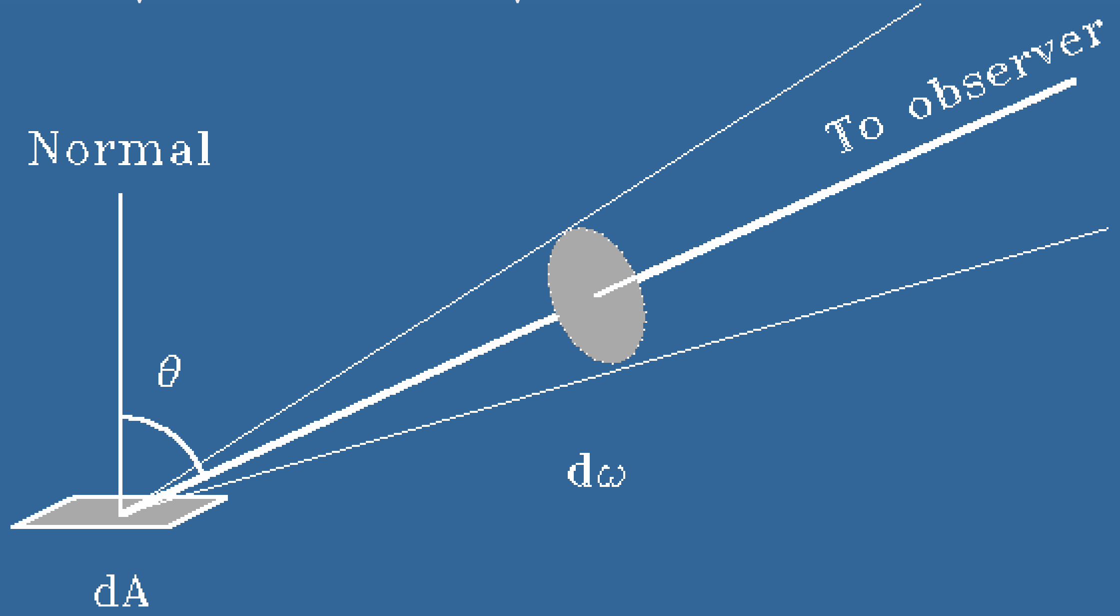
Total Recall (Lecture 1)

Radiative transfer: main concepts and definitions



$$\left.\begin{aligned} dE_\nu &= I_\nu \, dA \, d\omega \, dt \, d\nu \\ dE_\lambda &= I_\lambda \, dA \, d\omega \, dt \, d\lambda \end{aligned}\right\} \text{intensity}$$

$I_\lambda \, d\lambda = I_\nu d\nu$, units I_ν [erg/(s·cm^2·rad^2·Hz)]



Useful quantities

Mean intensity: $$J_\nu = \frac{1}{4\pi} \oint I_\nu \, d\omega$$

Flux: $$F_\nu = \oint I_\nu \cos\theta \, d\omega$$

Absorption coefficient: $$dI_\nu = -k_\nu \rho I_\nu dx$$

units of k_ν [cm^2/g]

Emission coefficient: $$dI_\nu = j_\nu \rho dx$$

units of j_ν [erg/(s rad^2 Hz g)]

Optical depth: $d\tau_\nu = k_\nu \rho \, d x$, τ_ν is unitless!

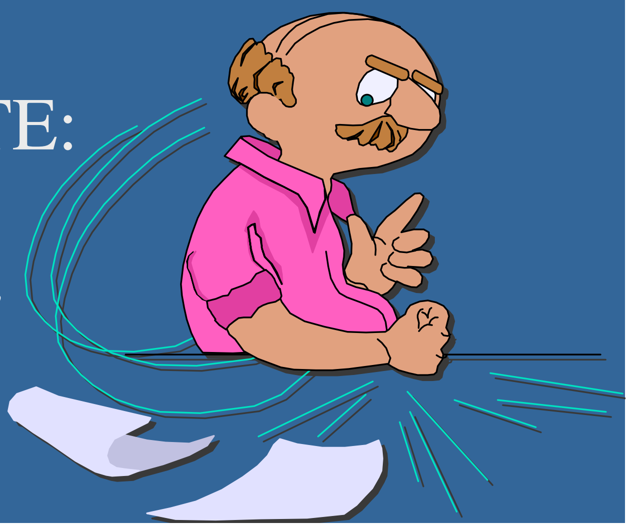
Source function: ratio of j_ν to k_ν

- Absorption and emission contain the "true" part (*energy is transferred between the kinetic energy of the gas and the radiation field*) and the scattering part (*energy of absorbed photon which is returned to the radiation field*).
- Radiation dominated gas: pure scattering.
- Collision dominated gas: pure absorption.
- In general case:

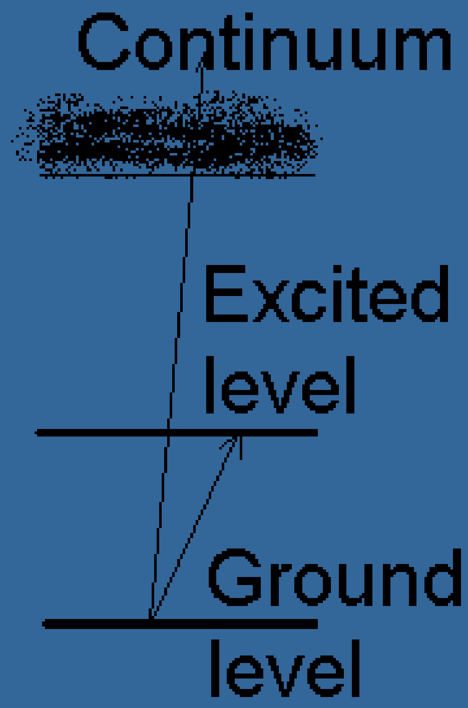
$$S_\nu = S_\nu^S + S_\nu^A$$

- For isotropic scattering and LTE:

$$S_\nu = \frac{\sigma_\nu}{k_\nu + \sigma_\nu} J_\nu + \frac{k_\nu}{k_\nu + \sigma_\nu} B_\nu$$



- One can distinguish 3 types of absorption processes:
- ***b-b*** - radiative transitions
  - collisional transitions
- ***b-f*** - ionization and recombination
- ***f-f*** -absorption/emission



- Radiative ***b-b*** transitions: absorption, spontaneous and stimulated emission.
- Collisional ***b-b*** transitions: excitation and de-excitation

Equation of radiative transfer:

$$\mathrm{d}\,I_\nu = -k_\nu \rho I_\nu\,\mathrm{d}\,x + j_\nu \rho\,\mathrm{d}\,x$$

or

$$\frac{\mathrm{d}\,I_\nu}{\mathrm{d}\,\tau_\nu} = -I_\nu + S_\nu$$

One can obtain formal solution:

$$I_\nu\left(\tau_\nu^{''}\right) = \int_{\tau_\nu^{'}}^{\tau_\nu^{''}} S_\nu(t)\exp\left[-\left(\tau_\nu^{''} - t\right)\right]\mathrm{d}\,t + I_\nu\left(\tau_\nu^{'}\right)\exp\left[-\left(\tau_\nu^{''} - \tau_\nu^{'}\right)\right]$$

Unfortunately, S_ν depends on I_ν!

Critical dependences

- Geometrical, angular, and frequency dependence of opacity k_ν and source function S_ν
- Dependence of the source function S_ν on the radiation field
- Number of absorbers (how many absorbers there is on a given energy level) depend on local physical conditions and radiation field
- Velocity distribution of the absorbers affects the frequency dependence of k_ν and S_ν

  Examples: think of some where one of the above has dramatic effects on radiation field

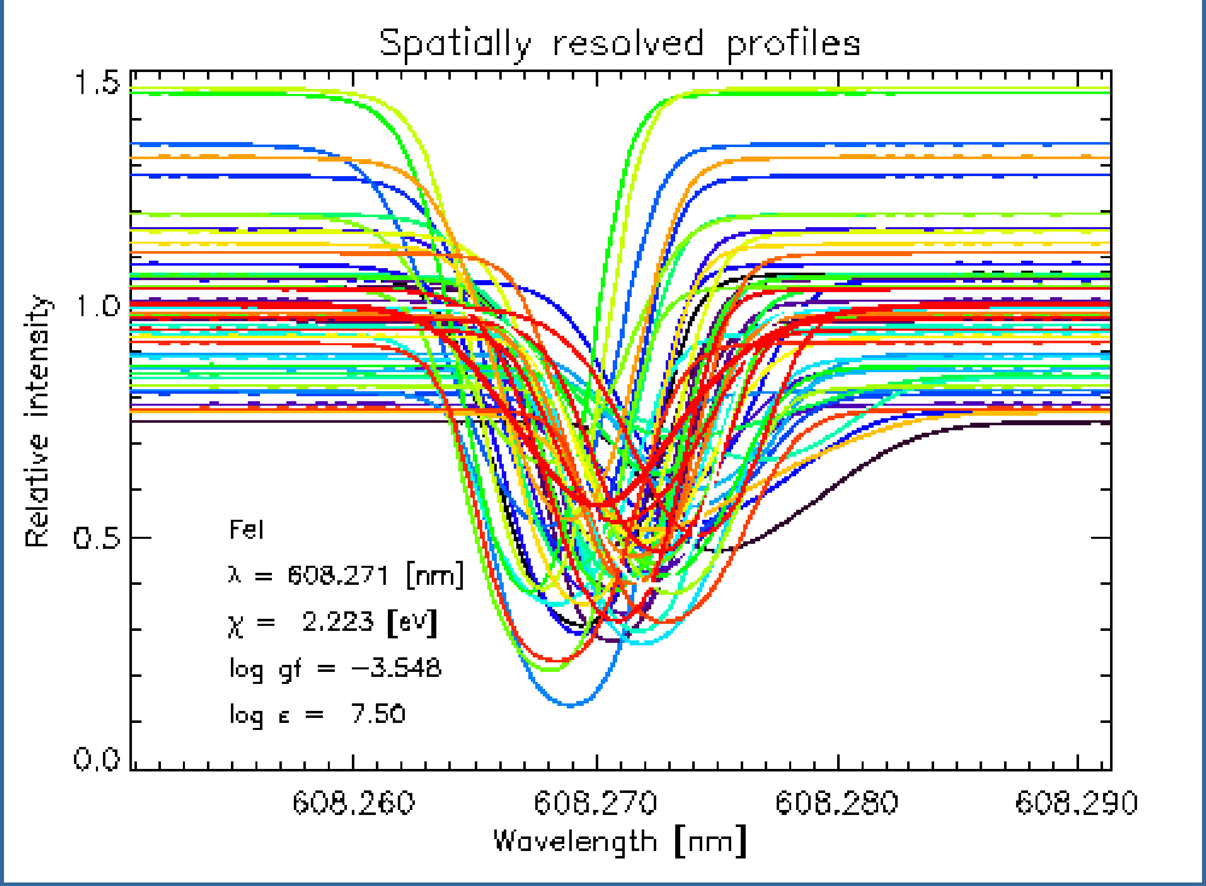
Approximations

- Geometry: static, plane-parallel or spherical medium
- Angular: isotropic radiation field
- Absorbers: Boltzmann level population, Saha ionization balance, Maxwellian velocity distribution
- Line shapes: identical absorption/emission profiles, Voigt profile
- Local Thermodynamical Equilibrium (LTE, how good is it?)

Examples

- Photospheres of solar-like stars (convection)
- Giant stars (spherical, anisotropic radiation field, giant convection cells)
- Stellar winds (complex geometry, velocity field, anisotropic radiation field, NLTE, dynamic)
- Gas clouds (LTE?, external radiation field, different T_{rad} and T_{gas}, presence of dust)

Solar convection and emerging spectra



Courtesy of Martin Asplund

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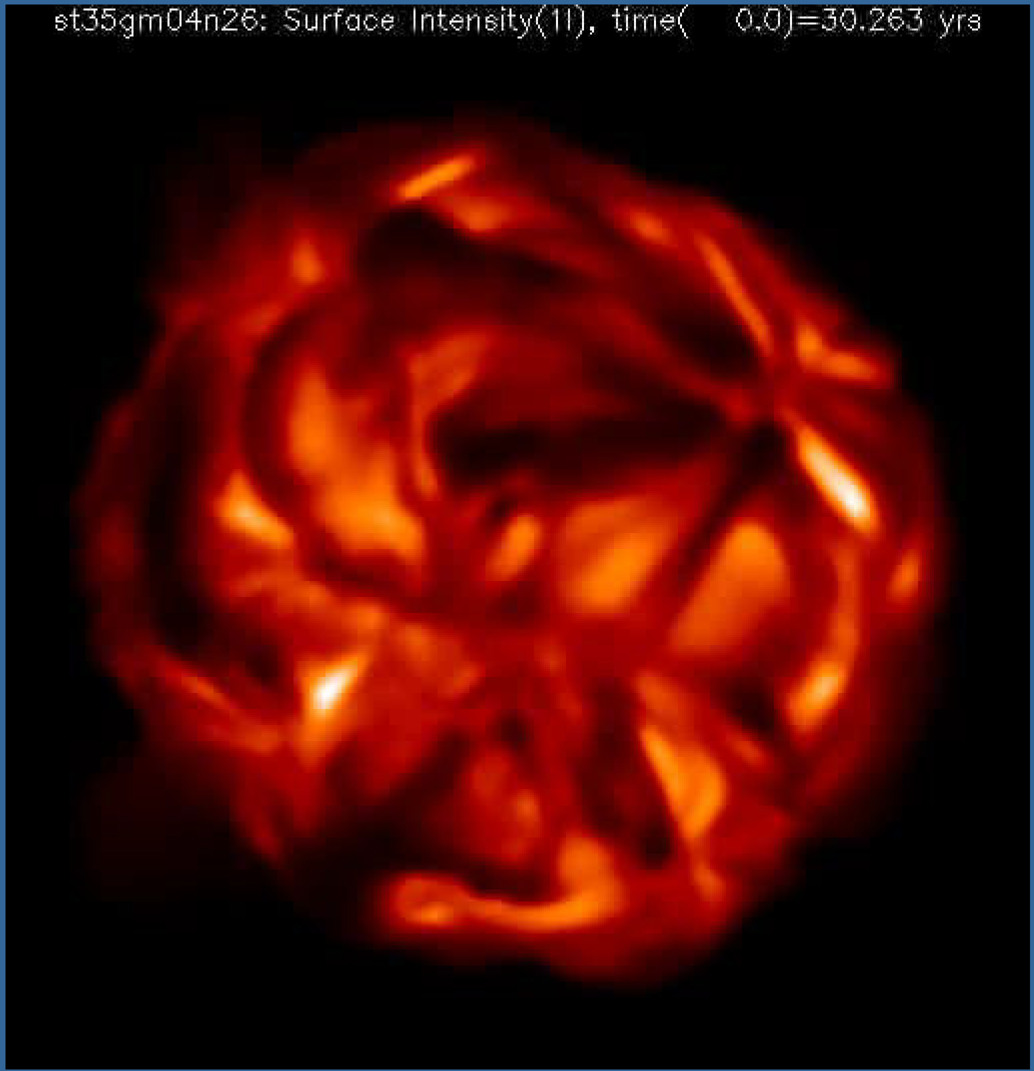
Convection on Betelgeuse



Courtesy of Bernd Freytag

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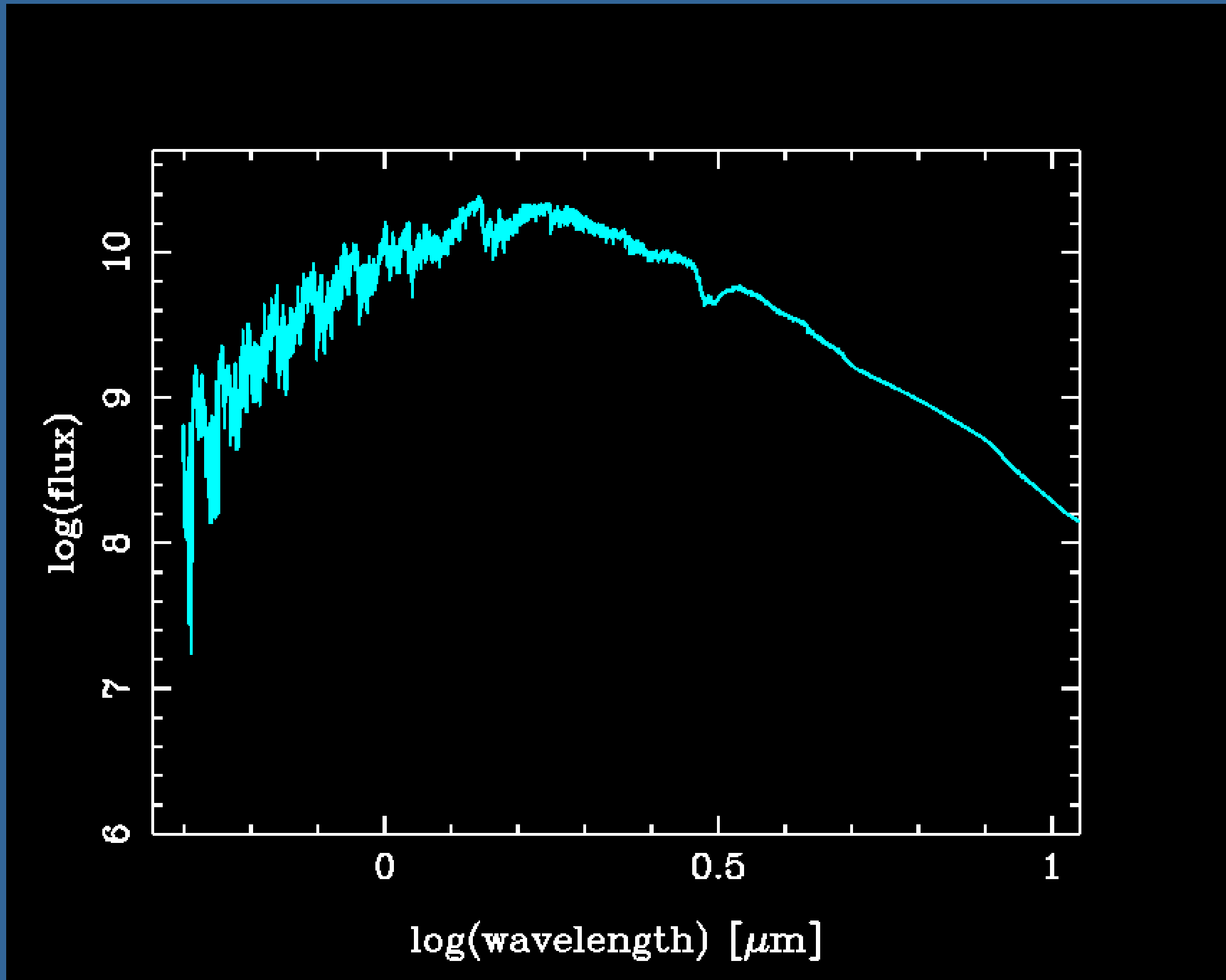
Dynamic spectra



Courtesy of Susanne Höfner

Examples

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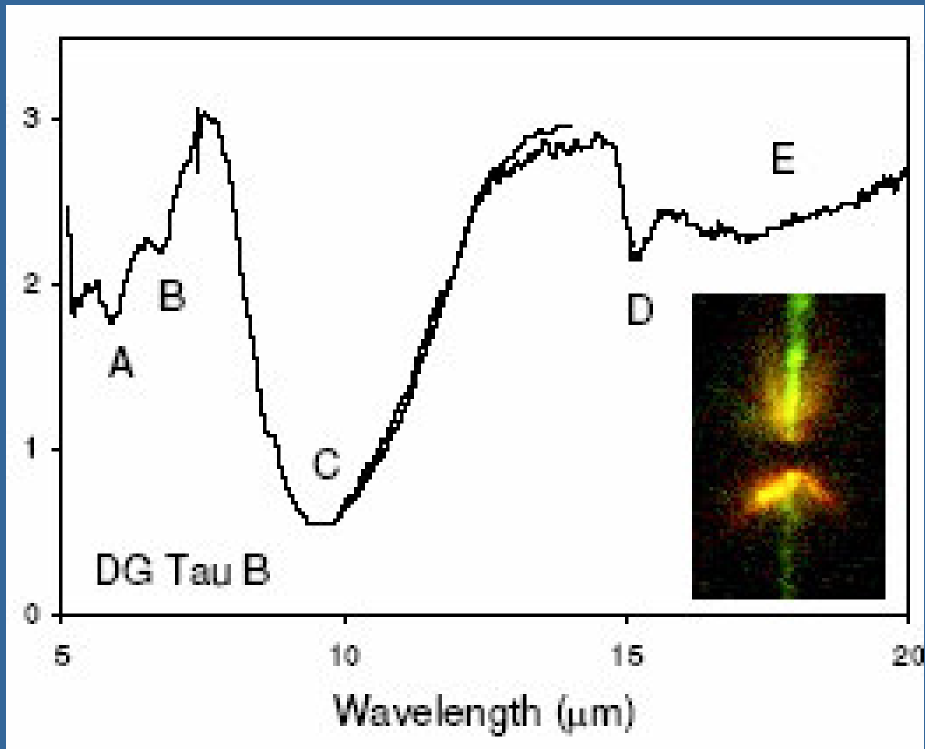
Protostars



A – water ice

B – methanol ice

C & E – amorphous silicates

D – carbon-dioxide ice

Watson et al.: 2004, Astrophysical Journal Supp Series 154, 391

Next part

- Math: first and second order ordinary differential equations, partial differential equations, boundary conditions, direct integration schemes, finite differences, convergence and stability, vector ODE. Gauss quadratures, solving systems of linear equations. Non-linear equations.
- Press et al. 1992, “Numerical Recipes: The Art of Scientific Computing”
  http://www.numerical-recipes.com/nronline_switcher.html

Total Recall

Math

System of Linear Equations (SLE):

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2 \\
\qquad\qquad\qquad\quad \ldots \qquad\qquad\quad \ldots \\
a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N
\end{array}
$$

$$A \cdot x = b$$

Two important algorithms: *Gauss-Jordan elimination* and *LU decomposition*

Gauss-Jordan Elimination

Simple rules:

i. Changing places of any two rows in A requires only a similar change in b.

ii. Replacing any row in A and b with a linear combination of itself and other rows does not change the solution.

iii. Interchanging two columns in A is equivalent to changing the sequence of x, therefore the solution must be sorted to get the original sequence.

No pivoting

$$
\begin{aligned}
a_{11}x_1+a_{12}x_2+ \quad\ldots &= b_1\\
a_{21}x_1+a_{22}x_2+ \quad\ldots &= b_2\\
a_{31}x_1+a_{32}x_2+ \quad\ldots &= b_3
\end{aligned}
$$

$$
\begin{aligned}
a_{11}x_1+a_{12}x_2+ \quad\ldots &=b_1\\
0+\quad (a_{22}-a_{12}a_{21}/a_{11})x_2+\ldots &=b_2-b_1a_{21}/a_{11}\\
0+\quad (a_{32}-a_{12}a_{31}/a_{11})x_2+\ldots &=b_3-b_1a_{31}/a_{11}
\end{aligned}
$$

$$
\begin{aligned}
a'_{11}x_1+a'_{12}x_2+ \quad\ldots &= b'_1\\
a'_{22}x_2+ \quad\ldots &= b'_2\\
a'_{32}x_2+ \quad\ldots &= b'_3
\end{aligned}
$$

Partial pivoting

$$a_{11}x_1+a_{12}x_2+ \quad \ldots = b_1$$
$$a_{21}x_1+a_{22}x_2+ \quad \ldots = b_2$$
$$a_{31}x_1+a_{32}x_2+ \quad \ldots = b_3$$

$k: \ |a_{k1}|=\max(|a_{j1}|)$ for $j=i,i+1,\ldots N$

$$a_{k1}x_1+a_{k2}x_2+ \quad \ldots =b_k$$
$$0+ \quad (a_{22}-a_{k2}a_{21}/a_{k1})x_2+\ldots =b_2-b_k a_{21}/a_{k1}$$
$$0+ \quad (a_{32}-a_{k2}a_{31}/a_{k1})x_2+\ldots =b_3-b_k a_{31}/a_{k1}$$

$$a'_{11}x_1+a'_{12}x_2+ \quad \ldots = b'_1$$
$$a'_{22}x_2+ \quad \ldots = b'_2$$
$$a'_{32}x_2+ \quad \ldots = b'_3$$

Full pivoting

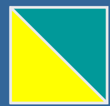
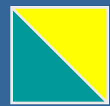
$$a_{11}x_1+a_{12}x_2+ \qquad \ldots = b_1$$
$$a_{21}x_1+a_{22}x_2+ \qquad \ldots = b_2$$
$$a_{31}x_1+a_{32}x_2+ \qquad \ldots = b_3$$

k,l: $|a_{kl}|=\max(|a_{nm}|)$ for n,m=i,i+1,… N

$$a_{k1}x_1+a_{k2}x_2+ \qquad \ldots =b_k$$
$$0+ \quad (a_{22}-a_{k2}a_{21}/a_{k1})x_2+\ldots =b_2-b_ka_{21}/a_{k1}$$
$$0+ \quad (a_{32}-a_{k2}a_{31}/a_{k1})x_2+\ldots =b_3-b_ka_{31}/a_{k1}$$

$$a'_{11}x_1+a'_{12}x_2+ \qquad \ldots = b'_1$$
$$a'_{22}x_2+ \qquad \ldots = b'_2$$
$$a'_{32}x_2+ \qquad \ldots = b'_3$$

LU decomposition

$A = L \cdot U$ where L and U are triangular matrices L:  U: 

$A \cdot x = (L \cdot U) \cdot x = L \cdot (U \cdot x) = b$

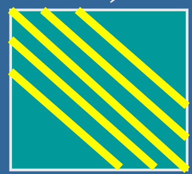
$L \cdot y = b$ and $U \cdot x = y$

Solving these systems is easy:
$y_1 = b_1 / L_{11}$; $y_2 = (b_2 - y_1 \cdot L_{21}) / L_{22}$ etc.

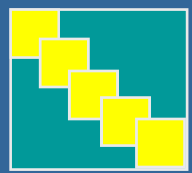
See Num. Rec. Section 2.3 on how to compute L and U.

Special matrices

- Tri-diagonal: forward and back-substitution, (no difference between Gauss-Jordan and LU decomposition schemes)
- Band-diagonal



- Block-diagonal



Iterative improvement of the solution:

$$A \cdot (x - x') = A \cdot x' - b$$